

Quadratic matings and ray connections

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Abstract

A topological mating is a map defined by gluing together the filled Julia sets of two quadratic polynomials. The identifications are visualized and understood by pinching ray-equivalence classes of the formal mating. For postcritically finite polynomials in non-conjugate limbs of the Mandelbrot set, classical results construct the geometric mating from the formal mating. Here families of examples are discussed, such that all ray-equivalence classes are uniformly bounded trees. Thus the topological mating is obtained directly in geometrically finite and infinite cases. On the other hand, renormalization provides examples of unbounded cyclic ray connections, such that the topological mating is not defined on a Hausdorff space.

There is an alternative construction of mating, when at least one polynomial is preperiodic: shift the infinite critical value of the other polynomial to a preperiodic point. Taking homotopic rays, it gives simple examples of shared matings. Sequences with unbounded multiplicity of sharing, and slowly growing preperiod and period, are obtained both in the Chebychev family and for Airplane matings. Using preperiodic polynomials with identifications between the two critical orbits, an example of mating discontinuity is described as well.

1 Introduction

Starting from two quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, construct the topological mating $P \amalg Q$ by gluing the filled Julia sets \mathcal{K}_p and \mathcal{K}_q . If there is a conjugate rational map f , this defines the geometric mating. These maps are understood by starting with the formal mating $g = P \sqcup Q$, which is conjugate to P on the lower half-sphere $|z| < 1$ and to Q on the upper half-sphere $|z| > 1$ of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: ray-equivalence classes consist of external rays of P and Q with complex conjugate angles, together with landing points in $\partial\mathcal{K}_p$ and $\partial\mathcal{K}_q$; collapsing these classes defines the topological mating. In the postcritically finite case, with p and q not in conjugate limbs of \mathcal{M} , either g or a modified version \tilde{g} is combinatorially equivalent and semi-conjugate to a rational map f [56, 10, 16, 25, 54]. So the topological mating exists and f is conjugate to it — it is a geometric mating.

In general both \mathcal{K}_p and \mathcal{K}_q contain pinching points and branch points with several rays landing together, so there are ray-equivalence classes consisting of subsequent

rays connecting points in $\partial\mathcal{K}_p$ and $\partial\mathcal{K}_q$ alternately. For rational angles, the landing pattern is understood combinatorially, and the identifications of periodic and preperiodic points can be determined. Consider the example of the 5-periodic p with the external angle $11/31$ and preperiodic q with angle $19/62$ in Figure 1: since q belongs to the $2/5$ -limb of \mathcal{M} , there are five branches of \mathcal{K}_q and five external rays at the fixed point α_q , which are permuted with rotation number $2/5$ by Q . Now p is chosen such that the complex conjugate angles land pairwise with another 5-cycle at the Fatou basins; the rays of Q corresponding to the latter angles land at endpoints, including the iterate $Q(q)$ of the critical value. So in the topological mating and in the geometric mating $f \cong P \amalg Q$, the point $Q(q)$ is identified both with α_q and with a repelling 5-cycle of P . Now the critical point 0 of f is 5-periodic, while $f^2(\infty)$ is fixed. The five components of the immediate attracting basin all touch at this fixed point with rotation number $2/5$, although they had disjoint closures in \mathcal{K}_p .

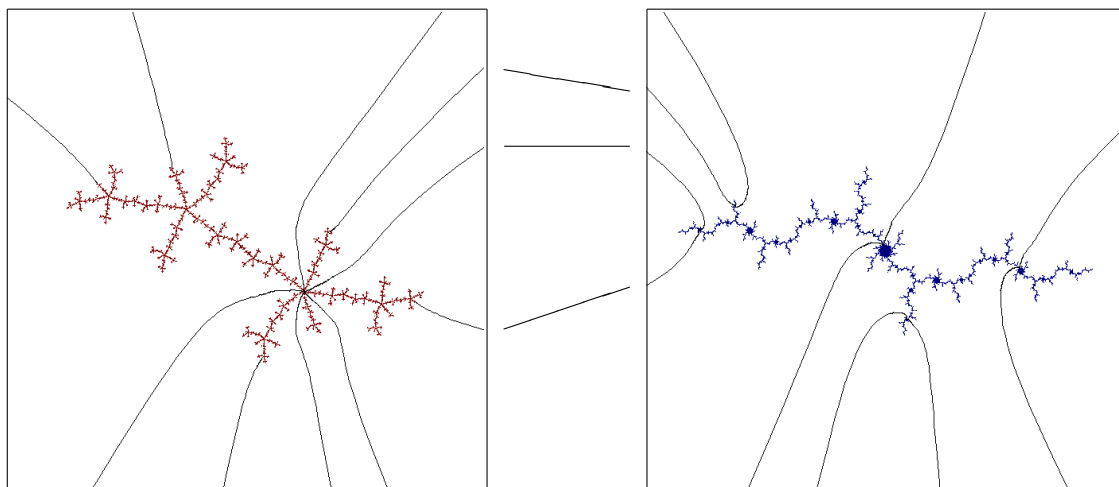


Figure 1: A formal mating $g = P \amalg Q$. The Julia set \mathcal{K}_p for the five-periodic center p corresponding to $\gamma_M(11/31)$ is shown on the right; the Misiurewicz Julia set \mathcal{K}_q with $q = \gamma_M(19/62)$ in the left image is rotated. (This does not change the set itself, but its external rays.) The ray connections and dynamics are discussed in the main text.

There are various ways to visualize the sets $\varphi_0(\mathcal{K}_p), \varphi_\infty(\mathcal{K}_q) \subset \widehat{\mathbb{C}}$ in the plane \mathbb{C} : instead of \mathcal{K}_q coming from ∞ , we may rotate the sphere such that \mathcal{K}_q is translated above or below \mathcal{K}_p , or to save space here, translated to the left or right and rotated.

In any case, $\varphi_0(\mathcal{R}_p(\theta))$ is connected with $\varphi_\infty(\mathcal{R}_q(-\theta))$; three connections are indicated between the two images. When discussing the combinatorics of a ray-equivalence class, we may avoid conjugation of several angles by assuming that $\mathcal{R}_p(\theta)$ connects to $\mathcal{R}_q(\theta)$, but to draw these rays without crossing, you would need to stack two sheets of paper.

Basic definitions and the geometry of ray-equivalence classes are discussed in **Section 2**. Simple examples of shared matings and of mating discontinuity are obtained in **Section 3**. The rational map f above belongs to the same one-parameter family as matings with the Chebychev polynomial, but it is not of this form. There are five other representations as a mating: take the Rabbit with rotation number $2/5$ for P and suitable preperiodic parameters q_1, \dots, q_5 for Q , which are related to the angles at $-\alpha_p$. More generally, we have $P \amalg Q_i = P \amalg Q_j$ for all p in the small satellite Mandelbrot set, since the rays at $-\alpha_p$ are homotopic with respect to the postcritical

set and so the precaptures are combinatorially equivalent. Taking higher rotation numbers gives shared matings with larger multiplicity. While it is obvious that a hyperbolic rational map has only a finite number of representations as a mating, this is not known in general when one or both of the critical points are preperiodic. Finiteness is shown here for Chebychev maps with one critical point periodic, and in [28] for Lattès maps. Examples with arbitrarily high multiplicity are obtained as well for matings of the Airplane with preperiodic polynomials; here preperiod and period are of the same order as the multiplicity, in contrast to the hyperbolic examples by Rees [46], where the period grows exponentially. — Simple ray connections can be used to define preperiodic matings with $f(0) = \infty$. This property is lost when preperiodic parameters converge to a parabolic parameter, confirming that mating is not jointly continuous. The mechanism is similar to geometrically infinite examples by Blé–Valdez–Epstein [5, 19], but here all maps are geometrically finite and matability does not require special arguments.

In general there is only a Cantor set of angles at the Hubbard tree $T_{\bar{q}} \subset \mathcal{K}_{\bar{q}}$, whose Hausdorff dimension is less than 1. If an open interval in the complement contains all angles on one side of the arc $[-\alpha_p, \alpha_p] \subset \mathcal{K}_p$, ray connections of the formal mating $P \sqcup Q$ are bounded explicitly, and the topological mating exists. This approach was used by Shishikura–Tan in a cubic example [55]; in the quadratic case it generalizes the treatment of $1/4 \amalg 1/4$ by Milnor [42] to large classes of examples. These include the mating of Airplane and Kokopelli, answering a question by Adam Epstein [9]: can the mating be constructed without employing the theorems of Thurston and Rees–Shishikura–Tan? See **Section 4**. Note however, that only the branched covering on the glued Julia sets is constructed here, not a conjugate rational map. On the other hand, the method applies to geometrically infinite parameters as well. Examples of irrational ray connections and an algorithm for finding long ray connections are discussed in addition. In **Section 5**, specific ray connections for polynomials from conjugate limbs are obtained, which are related to renormalization of one polynomial. These ray-equivalence classes accumulate on the Julia set, such that the quotient space is not Hausdorff.

— This is the second paper in a series on matings and other applications of the Thurston Algorithm [10, 16, 25]:

- **The Thurston Algorithm for quadratic matings [27].** The Thurston Algorithm for the formal mating is implemented by pulling back a path in moduli space; an alternative initialization by a repelling-preperiodic capture is discussed as well. When the Thurston Algorithm diverges in ordinary Teichmüller space due to postcritical identifications, it still converges on the level of rational maps and colliding marked points — it is not necessary to implement the essential mating by encoding ray-equivalence classes numerically. The proof is based on the extended pullback map on augmented Teichmüller space constructed by Selinger [50, 51].
- **Quadratic matings and ray connections [the present paper].**
- **Quadratic matings and Lattès maps [28].** Lattès maps of type $(2, 2, 2, 2)$ or $(2, 4, 4)$ are represented by matings in basically nine, respectively three, different ways. This is proved from combinatorics of polynomials and ray-equivalence classes. The Shishikura Algorithm relates the topology of the

formal mating to the multiplier of the corresponding affine map on a torus. The slow mating algorithm diverges in certain cases: while the expected collisions are happening, a neutral eigenvalue from the one-dimensional Thurston Algorithm persists, producing an attracting center manifold in moduli space. (Joint work with Arnaud Chéritat.) Twisted Lattès maps are discussed as well, and the Hurwitz equivalence between quadratic rational maps with the same ramification portrait is constructed explicitly, complementing the approach related to the moduli space map by Sarah Koch [32].

- **Slow mating and equipotential gluing [14], jointly with Arnaud Chéritat.** Equipotential gluing is an alternative definition of mating, not based on the Thurston Algorithm. Equipotential lines of two polynomials are glued to define maps between spheres, and the limit of potential 0 is considered. The initialization of the slow mating algorithm depends on an initial radius R ; when $R \rightarrow \infty$, slow mating is shown to approximate equipotential gluing. The visualization in terms of holomorphically moving Julia sets and their convergence is discussed and related to the notion of conformal mating.
- **Quadratic captures and anti-matings [30].** The slow Thurston Algorithm is implemented for captures and for anti-matings as well. The latter means that two planes or half-spheres are mapped to each other by quadratic polynomials, and the filled Julia sets of two quartic polynomials are glued together. There are results analogous to matings, but a complete combinatorial description does not exist due to the complexity of even quartic polynomials. For specific families of quadratic rational maps, the loci of mating, anti-mating, and captures are obtained numerically.
- **The Thurston Algorithm for quadratic polynomials [31].** The slow Thurston Algorithm is implemented for several kinds of Thurston maps giving quadratic polynomials. These include a spider algorithm with a path instead of legs, Dehn twisted polynomials, moving the critical value by recapture or precapture, and tuning. Using the Selinger results on removable obstructions, the spider algorithm is shown to converge in the obstructed case of satellite Misiurewicz points as well. Recapture surgery is related to internal addresses, and used to discuss a specific example of twisted polynomials.

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2 Mating: definitions and basic properties

After recalling basic properties of quadratic polynomials and matings, the geometry of rational and irrational ray-equivalence classes is described, generalizing an observation by Sharland [53]. Repelling-preperiodic captures are considered as an alternative construction of matings; the proof was given in [27], using the relation between ray-equivalence classes and Thurston obstructions from [56, 54].

2.1 Polynomial dynamics and combinatorics

For a quadratic polynomial $f_c(z) = z^2 + c$, the filled Julia set \mathcal{K}_c contains all points z with $f_c^n(z) \not\rightarrow \infty$. It is connected, if and only if the critical point $z = 0$ does not escape, and then the parameter c belongs to the Mandelbrot set \mathcal{M} by definition. A **dynamic ray** $\mathcal{R}_c(\theta)$ is the preimage of a straight ray with angle $2\pi\theta$ under the Boettcher conjugation $\Phi_c : \widehat{\mathbb{C}} \setminus \mathcal{K}_c \rightarrow \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. For rational θ , the rays and landing points are periodic or preperiodic under f_c , since $f_c(\mathcal{R}_c(\theta)) = \mathcal{R}_c(2\theta)$. If two or more periodic rays land together, this defines a non-trivial orbit portrait; it exists if and only if the parameter c is at or behind a certain root [48, 41]. There are analogous parameter rays with rational angles $\mathcal{R}_M(\theta)$ landing at roots and Misiurewicz points; the angles of a root are characteristic angles from the orbit portrait. In particular, the k/r -limb and wake of the main cardioid are defined by two parameter rays with r -periodic angles, and for the corresponding parameters c , the fixed point $\alpha_c \in \mathcal{K}_c$ has r branches and external angles permuted with rotation number k/r . Denote landing points by $z = \gamma_c(\theta) \in \partial\mathcal{K}_c$ and $c = \gamma_M(\theta) \in \partial\mathcal{M}$, respectively. f_c is **geometrically finite**, if it is preperiodic, hyperbolic, or parabolic.

Proposition 2.1 (Douady Magic Formula, Blé)

Suppose $\theta \in [0, 1/3]$ is an external angle of the main cardioid, then $\Theta = 1/2 + \theta/4 \in [1/2, 7/12]$ is an external angle of the real axis $\mathcal{M} \cap \mathbb{R}$.

Proof: According to [11], the orbit of θ under doubling is confined to $[\theta/2, (1 + \theta)/2]$. Now taking a suitable preimage shows that the orbits of θ and Θ never enter $((\theta + 1)/4, (\theta + 2)/4) \supset (1 - \Theta, \Theta)$, so Θ is combinatorially real: it defines a unique real parameter c by approximation, and the parameter ray $\mathcal{R}_M(\Theta)$ accumulates at a fiber [47] intersecting the real line in c . Blé [4] has shown that f_c is strongly recurrent but not renormalizable, so the fiber is trivial and the ray actually lands, $c = \gamma_M(\Theta)$. ■

2.2 Topological mating and geometric mating

For parameters $p, q \in \mathcal{M}$ with locally connected Julia sets, define the **formal mating** $g = P \sqcup Q$ of the quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ as follows: $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a branched covering with critical points 0 and ∞ , and normalized such that $g(z) = z^2$ for $|z| = 1$. On the lower and upper half-spheres, g is topologically conjugate to P and Q by homeomorphisms φ_0 and φ_∞ , respectively. An **external ray** $\mathcal{R}(\theta)$ of g is the union of $\varphi_0(\mathcal{R}_p(\theta))$ and $\varphi_\infty(\mathcal{R}_q(-\theta))$ plus a point on the equator; each ray connects a point in $\varphi_0(\mathcal{K}_p)$ to a point in $\varphi_\infty(\mathcal{K}_q)$. A **ray-equivalence class** is a maximal connected set consisting of rays and landing points. Collapsing all classes to points may define a Hausdorff space homeomorphic to the sphere; then the map corresponding to g is a branched covering again [44], which defines the **topological mating** $P \sqcup Q$ up to conjugation. By the identifications, periods may be reduced and different orbits meet. We are interested in a rational map f conjugate to the topological mating, and we shall speak of “the” geometric mating when the following normalization is used. Note however, that uniqueness is not obvious when the polynomials are not geometrically finite, in particular if there is a locally connected Julia set carrying an invariant line field.

Definition 2.2 (Normalization of the geometric mating)

Suppose the topological mating $P \amalg Q$ is topologically conjugate to a quadratic rational map F , and the conjugation ψ is conformal in the interior of the filled Julia sets. Then the geometric mating exists and it is Möbius conjugate to F .

The **geometric mating** $f \cong P \amalg Q$ is normalized such that ψ maps the critical point of P to 0, the critical point of Q to ∞ , and the common β -fixed point to 1. If the latter condition is dropped, then f is affine conjugate to the geometric mating, and we shall write $f \simeq P \amalg Q$.

Sometimes it is convenient to write $p \amalg q$ or $\theta_p \amalg \theta_q$ for $P \amalg Q$; here a periodic angle is understood to define a center, not a root. In the postcritically finite case, the geometric mating is constructed using Thurston theory as follows:

Theorem 2.3 (Rees–Shishikura–Tan)

Suppose P and Q are postcritically finite quadratic polynomials, not from conjugate limbs of the Mandelbrot set. Then the geometric mating $f \cong P \amalg Q$ exists.

Idea of the **proof**: The formal mating $g = P \sqcup Q$ is a postcritically finite branched covering, a Thurston map. So it is combinatorially equivalent to a rational map, if and only if it is unobstructed, excluding type $(2, 2, 2, 2)$ here [28]. According to Rees–Shishikura–Tan, all obstructions are Lévy cycles converging to ray-equivalence classes under iterated pullback [56]. See the example in Figure 3 of [27]. In the case of non-conjugate limbs, these obstructions are removed by collapsing postcritical ray-equivalence trees, which defines an unobstructed essential mating \tilde{g} . Now the Thurston Theorem [10, 16, 25] produces a rational map f equivalent to g or \tilde{g} , respectively, unique up to normalization. By iterating a suitable equivalence, a semi-conjugation from g to f is obtained [54], which collapses all ray-equivalence classes to points. So f is conjugate to the topological mating $P \amalg Q$. ■

Conjecture 2.4 (Quadratic mating)

For quadratic polynomials P and Q with locally connected Julia sets, the geometric mating exists, unless p and q are in conjugate limbs of the Mandelbrot set.

Originally, it was expected that mating depends continuously on the polynomials [39]; various counterexamples by Adam Epstein [19, 9] are discussed in Section 3.5, and a simple new counterexample is given. — The geometric mating is known to exist in the following quadratic cases:

- In the postcritically finite situation, Conjecture 2.4 was proved in [56, 54], cf. Theorem 2.3. In this case, the geometric mating exists, whenever the topological mating does. See [44, 14] for various notions of conformal mating.
- Suppose P and Q are hyperbolic quadratic polynomials, and denote the corresponding centers by p_0 and q_0 , let $f_0 \cong P_0 \amalg Q_0$. Now P_0 is quasiconformally conjugate to P in a neighborhood of the Julia set $\mathcal{J}_{p_0} = \partial\mathcal{K}_{p_0}$, analogously for Q_0 , and there is a rational map f with the corresponding multipliers, such that f_0 is quasiconformally conjugate to f in a neighborhood of \mathcal{J}_{f_0} . The conjugations of polynomials respect the landing of dynamic rays, so the semi-conjugations from P_0 and Q_0 to f_0 define new semi-conjugations from P and Q to f in neighborhoods of the Julia sets. Using conformal conjugations to

Blaschke products on the immediate basins, the required semi-conjugations from $\mathcal{K}_p \sqcup \mathcal{K}_q \rightarrow \widehat{\mathbb{C}}$ are constructed, and $f \cong P \amalg Q$ is a geometric mating. The same argument works when one polynomial is hyperbolic and the other one is preperiodic.

- A geometrically finite quadratic polynomial is preperiodic, hyperbolic, or parabolic. Haïssinsky–Tan have constructed all matings of geometrically finite polynomials from non-conjugate limbs [22]: when parabolic parameters are approximated radially from within hyperbolic components, the geometric matings converge. The proof is based on distortion control techniques by Cui. On the other hand, when two parabolic parameters are approximated tangentially, mating may be discontinuous; see [19, 9] and Section 3.5.
- For quadratic polynomials having a fixed Siegel disk of bounded type, Yampolsky–Zakeri [61] construct the geometric mating when the multipliers are not conjugate, and obtain the mating of one Siegel polynomial with the Chebychev polynomial in addition. The proof combines Blaschke product models, complex a priori bounds, and puzzles with bubble rays.
- Suppose θ defines a parameter p with a Siegel disk of bounded type and consider the real parameter q with angle $\Theta = 1/2 + \theta/4$ defined in Proposition 2.1, which is strongly recurrent. The geometric mating $f \cong P \amalg Q$ exists according to Blé-Valdez [5].
- Denote the family of quadratic rational maps $f_a(z) = (z^2 + a)/(z^2 - 1)$ with a superattracting 2-cycle by V_2 . It looks like a mating between the Mandelbrot set \mathcal{M} and the Basilica Julia set \mathcal{K}_B , both truncated between the rays with angles $\pm 1/3$. Capture components correspond to Fatou components of the Basilica. Large classes of maps in V_2 are known to be matings of quadratic polynomials with the Basilica, by work of Luo, Aspenberg–Yampolsky, Dudko, and Yang [33, 1, 17, 62]. The basic idea is to construct puzzle-pieces with bubble rays both in the dynamic plane and in the parameter plane. This approach does not seem to generalize to V_3 , because Rabbit matings may be represented by Airplane matings as well.
- When p is periodic and \bar{q} shares an angle with a boundary point of a preperiodic Fatou component, the geometric mating is constructed by regluing a capture according to Mashanova–Timorin [35].
- For large classes of geometrically finite and infinite examples, Theorem 4.2 shows that ray-equivalence classes are uniformly bounded trees. So the topological mating exists according to Epstein [44], but the geometric mating is not constructed here.

In higher degrees, a topological mating $P \amalg Q$ may exist when there is no geometric mating. An example with periodic cubic polynomials is discussed in [55]. Other examples are obtained from expanding Lattès maps: choose a 2×2 integer matrix A with trace t and determinant d satisfying $0 < t-1 < d < t^2/4$, e.g., $t = d = 5$. This defines a Thurston map g of type $(2, 2, 2, 2)$ with degree d . Now g^n is expanding and not equivalent to a rational map, since the eigenvalues of A^n are real > 1 and distinct [16, 25, 28]. But according to [37], g^n is a topological mating for large n .

2.3 Ray connections and ray-equivalence classes

For the mating of quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ with locally connected Julia sets, rays and ray-equivalence classes are defined in terms of the formal mating $g = P \sqcup Q$. A ray connection is an arc within a ray-equivalence class. The length of an arc or loop is the number of rays involved, and the diameter of a ray-equivalence class is the greatest distance with respect to this notion of length. We shall discuss the structure of ray-equivalence classes in detail for various examples, and show existence of the topological mating in certain cases. By the Moore Theorem [44], all ray-equivalence classes must be trees and the ray-equivalence relation must be closed. For this the length of ray connections will be more important than the number of rays and landing points in a ray-equivalence class: there is no problem when, e.g., branch points with an increasing number of branches converge to an endpoint, since the angles will have the same limit. The following results are proved in Propositions 4.3 and 4.12 of [44]:

Proposition 2.5 (Ray connections and matability, Epstein)

Consider ray-equivalence classes for the formal mating $g = P \sqcup Q$ of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, with \mathcal{K}_p and \mathcal{K}_q locally connected.

1. If all classes are trees and uniformly bounded in diameter, the topological mating $P \sqcup Q$ exists as a branched covering of the sphere.
2. If there is an infinite or a cyclic ray connection, the topological mating does not exist.

Note that there is no statement about non-uniformly bounded trees. For Misiurewicz matings having a pseudo-equator, Meyer [37] has shown that ray-equivalence classes are bounded uniformly in size; hence the diameters are bounded uniformly as well. Theorem 4.2 gives topological matings $P \sqcup Q$, where all ray-equivalence classes are bounded uniformly in diameter, but they need not be bounded in size; see Example 4.3. The following description of ray-equivalence classes can be given in general, speaking of connections between $\partial\mathcal{K}_p$ and $\partial\mathcal{K}_{\bar{q}}$ according to Figure 1:

Proposition 2.6 (Shape of ray-equivalence classes, following Sharland)

Consider rational and irrational ray-equivalence classes for the formal mating $g = P \sqcup Q$ of quadratic polynomials, with \mathcal{K}_p and \mathcal{K}_q locally connected.

1. Any branch point of a ray-equivalence class is a branch point of \mathcal{K}_p or $\mathcal{K}_{\bar{q}}$. Thus it is precritical, critical, preperiodic, or periodic. So with countably many exceptions, all ray-equivalence classes are simple arcs (finite or infinite), or simple loops.
2. Suppose the periodic ray-equivalence class C is a finite tree, then all the angles involved are rational of the same ray period m . Either C is an arc and m -periodic as a set, or it contains a unique point z of period $m' = m/r$ with $r \geq 2$ branches. Then z is the only possible branch point of C , so C is a topological star when $r \geq 3$.
3. Suppose that the topological mating $P \sqcup Q$ exists. Then only critical and precritical ray-equivalence classes may have more than one branch point. More precisely, we have the following cases:
 - a) Both P and Q are geometrically finite. Then irrational ray-equivalence classes of g are finite arcs, and rational ray-equivalence classes may have at most seven branch points.

b) *Precisely one of the two polynomials is geometrically finite. Then irrational classes have at most one branch point, and rational classes may have up to three.*

c) *Both polynomials are geometrically infinite. Then irrational classes have at most three branch points, and rational classes have at most one.*

Item 2 was used by Sharland [52, 53] to describe hyperbolic matings with cluster cycles. It is employed in Sections 4.3 and 6 of [28] to classify matings with orbifold of essential type $(2, 2, 2, 2)$, and here in Section 3.3.

Proof: 1. Since the rays themselves are not branched, the statement is immediate from the No-wandering-triangles Theorem [58, 48] for branch points of quadratic Julia sets.

2. Rational rays landing together have the same preperiod and ray period, and only rational rays land at periodic and preperiodic points of a locally connected Julia set. So they never land together with irrational rays. Ray-equivalence classes are mapped homeomorphically or as a branched cover. If a finite tree C satisfies $g^{m'}(C) \cap C \neq \emptyset$ with minimal $m' \geq 1$, we have $g^{m'}(C) = C$ in fact, and C does not contain a critical point. Since $g^{m'}$ is permuting the points and rays of C , there is a minimal $m \geq m'$, such that g^m is fixing all points and rays, and all angles are m -periodic. Suppose first that C contains a branch point z with $r \geq 3$ branches. It is of satellite type, so its period is $m/r \geq m'$, and the r branches are permuted transitively by $g^{m/r}$. Thus all the other points are m -periodic, and they cannot be branch points, because the first return map would not permute their branches transitively. So $m' = m/r$. On the other hand, if C is an arc, then $g^{m'}$ is either orientation-preserving and $m = m'$, or orientation-reversing and $m = 2m'$. In the latter case, the number of rays must be even, since each point is mapped to a point in the same Julia set, and the point in the middle has period $m' = m/2$.

3) A periodic ray-equivalence class may contain a single branch point according to item 2. In case a) a preperiodic class may contain two postcritical points (from different polynomials), and we have a pullback from critical value to critical point twice. Each time the number of branch points may be doubled, and a new branch point be created. This can happen only once in case b) and not at all in case c). On the other hand, an irrational ray-equivalence class C may contain only critical and precritical branch points, and this can happen only when the corresponding polynomial is geometrically infinite. Some image of C contains postcritical points instead of (pre-)critical ones, and it can contain only one postcritical point from each polynomial, since it would be periodic otherwise. So pulling it back to C again gives at most three branch points. Note that an irrational periodic class would be infinite or a loop, contradicting the assumption of matability. ■

2.4 Matings as repelling-preperiodic captures

A Thurston map may be defined by shifting a critical value to a preperiodic point along a path [45]:

Proposition 2.7 (and definition)

Suppose P is a postcritically finite quadratic polynomial and $z_1 \in \mathcal{K}_P$ is preperiodic and not postcritical. Let the new postcritical set be $P_g = P_P \cup \{P^n(z_1) \mid n \geq 0\}$.

Consider an arc C from ∞ to z_1 not meeting another point in P_g and choose a homeomorphism φ shifting ∞ to z_1 along C , which is the identity outside of a sufficiently small neighborhood of C . Then:

- $g = \varphi \circ P$ is well-defined as a quadratic Thurston map with postcritical set P_g . It is a **capture** if z_1 is eventually attracting and a **precapture** in the repelling case.
- The combinatorial equivalence class of g depends only on the homotopy class of the arc C .

See also the discussion of a possible numerical implementation of the Thurston Algorithm in [27]. Motivated by remarks of Rees and Mashanova–Timorin [35], the following result provides an alternative construction of quadratic matings in the non-hyperbolic case; see the proof of Theorem 6.3 in [27]:

Theorem 2.8 (Matings as precaptures, following Rees)

Suppose P is postcritically finite and θ is preperiodic, such that $q = \gamma_M(-\theta)$ is not in the conjugate limb and $z_1 = \gamma_p(\theta) \in \partial\mathcal{K}_p$ is not postcritical. Then the precapture $g_\theta = \varphi_\theta \circ P$ along $\mathcal{R}_p(\theta)$ is combinatorially equivalent or essentially equivalent to the geometric mating f defined by $P \amalg Q$.

3 Mating as a map between parameter spaces

Mating provides a partial map from $\mathcal{M} \times \mathcal{M}$ to the moduli space of quadratic rational maps. This map is neither surjective, injective, nor continuous. The characterization of matings in terms of equators and pseudo-equators by Thurston–Wittner and Meyer is discussed in Section 3.1. Old and new examples of shared matings are described in Section 3.2, and particular sequences with arbitrarily high multiplicity are obtained in Sections 3.3 and 3.4. Epstein has given various examples of mating discontinuity, which are described in Section 3.5, and a simple new construction is presented.

3.1 Characterization of matings

Hyperbolic quadratic rational maps f are classified as follows according to Rees [45] and Milnor [39]:

B or II is **bitransitive**: both critical points are in the same cycle of Fatou components but not in the same component.

C or III is a **capture**: one critical point is in a strictly preperiodic Fatou component.

D or IV has **disjoint** cycles of Fatou components.

E or I is **escaping**: both critical orbits converge to a fixed point within the only Fatou component.

Now each hyperbolic component of type B, C, D contains a unique postcritically finite map up to normalization, but there is no such map of type E. While hyperbolic anti-matings may be of type B, C, or D [30], every hyperbolic mating is of type D. The converse is false according to Ben Wittner [60]:

Example 3.1 (Wittner)

There is a unique real quadratic rational map of the form $f_w(z) = (z^2 + a)/(z^2 + b)$, such that 0 is four-periodic and ∞ is three-periodic; approximately $a = -1.3812$ and $b = -0.3881$. This map is not a geometric mating of quadratic polynomials.

Proof: Any mating $f \simeq P \sqcup Q$ has this branch portrait, if and only if P is four-periodic and Q is three-periodic. Wittner determined all combinations numerically and found them to be different from f_w . Alternatively, show combinatorially that all of these matings have periodic Fatou components with common boundary points; this is obvious when P or Q is of satellite type. Otherwise P or \bar{P} is the Kokopelli at $\gamma_M(3/15)$ and Q is the Airplane — then four-periodic Fatou components are drawn together pairwise by ray-connections through the two-cycle of the Airplane. On the other hand, for f_w no closed Fatou components in the same cycle meet, since the critical orbits are ordered cyclically as $z_3 < w_2 < z_0 < z_2 < w_1 < z_1 < w_0$ on $\mathbb{R} \cup \{\infty\}$. The Julia set \mathcal{J}_w is a Sierpinski carpet in fact [39]. ■

The characterization of matings by an equator is a folk theorem going back to Thurston; it was proved in [60, 38] under similar assumptions. Statement and proof require some standard notions from Thurston theory, see [10, 16, 25, 27].

Theorem 3.2 (Thurston–Lévy–Wittner)

Suppose f is a postcritically finite rational map of degree $d \geq 2$. Then f is combinatorially equivalent to a formal mating $g = P \sqcup Q$, if and only if it has an **equator** γ : a simple closed curve with the property that $\gamma' = f^{-1}(\gamma)$ is connected and homotopic to γ relative to the postcritical set, traversed in the same direction.

Proof: By construction, a formal mating $g = P \sqcup Q$ has the equator S^1 . So if f is combinatorially equivalent to g , with $\psi_0 \circ g = f \circ \psi_1$, then $\gamma = \psi_0(S^1)$ is homotopic to $\gamma' = f^{-1}(\gamma) = \psi_1(S^1)$. Conversely, when f has an equator, it is equivalent to a Thurston map \hat{g} with $\hat{g}(z) = z^d$ for $z \in S^1$. So \hat{g} is a formal mating of two topological polynomials \hat{P} and \hat{Q} . Suppose \hat{P} is obstructed, thus f is obstructed as well, then it would be a flexible Lattès map with four postcritical points. Now \hat{P} and \hat{Q} together have six postcritical points; since \hat{P} has at least four, \hat{Q} has at most two, so \hat{Q} and f have a periodic critical point. But Lattès maps have only preperiodic critical points, so \hat{P} and \hat{Q} are unobstructed in any case. By the Thurston Theorem, there are equivalent polynomials P and Q , which are determined uniquely by requiring them monic, centered, and with suitable asymptotics of the 0-ray under the equivalence. Now \hat{g} is equivalent to the formal mating $g = P \sqcup Q$. ■

Remark 3.3 (Equator and pseudo-equator)

1. Suppose $f \cong P \sqcup Q$ is a postcritically finite geometric mating. If f is hyperbolic, it is combinatorially equivalent to the formal mating $g = P \sqcup Q$, so it has an equator. If f is not hyperbolic, there may be identifications from postcritical ray-equivalence classes, such that g is obstructed and f is combinatorially equivalent to an essential mating \tilde{g} . Then f does not have an equator corresponding to this representation as a mating.

2. When P and Q have only preperiodic critical points, the essential mating \tilde{g} and the geometric mating f may have a **pseudo-equator**, which passes through all postcritical points; see [37, 38] for the definition. The equator of g is deformed to a pseudo-equator of \tilde{g} , if and only if there are at most direct ray connections between

postcritical points. Conversely, when f has a pseudo-equator γ , each pseudo-isotopy from γ to $f^{-1}(\gamma)$ determines a pair of polynomials P, Q with $f \simeq P \amalg Q$.

3. A Thurston map g is **expanding**, if there is a curve C through the postcritical points, such that its n -th preimages form a mesh with maximal diameters going to 0. See [7, 23, 3] for other notions of expansion. According to [37], some iterate g^n has a pseudo-equator and it is equivalent to a topological mating. A finite subdivision rule may be used to define an expanding map [12]; for an essential mating with a pseudo-equator, Wilkerson [59] constructs a subdivision rule from the Hubbard trees.

3.2 Shared matings

A shared mating is a geometric mating with different representations, $P_1 \amalg Q_1 \simeq f \simeq P_2 \amalg Q_2$ with $P_1 \neq P_2$ or $Q_1 \neq Q_2$. There are the following examples of shared matings, and techniques for constructing them:

- Wittner [60] introduced the notion of shared matings and discussed them for V_3 in particular. A simple example is given by the geometric mating of Airplane and Rabbit, which is affine conjugate to the geometric mating of Rabbit and Airplane, $A \amalg R \simeq R \amalg A$. (Moreover, it is conjugate to a symmetric map, which is not a self-mating.) Since the two polynomials are interchanged, this example is called the Wittner flip. It can be explained by finding two different equators, which has a few generalizations:
- Exall [20] constructs pairs of polynomials P, Q with $P \amalg R \simeq Q \amalg A$ from a second equator. Using symbolic dynamics, this can be done algorithmically.
- Rees [46] uses symbolic dynamics again to obtain unboundedly shared Airplane matings. The period grows exponentially with the multiplicity.
- Denote the rabbit of rotation number k/n by R . There are $n - 2$ primitive hyperbolic polynomials Q of period n , such that \overline{Q} has a characteristic angle from the cycle of α_r . Then the rational map $f \cong R \amalg Q$ has a cluster cycle: both n -cycles of Fatou components have a common boundary point, which is a fixed point corresponding to α_r . Tom Sharland [52, 53] has shown that f is determined uniquely by the rotation number and the relative displacement of the critical orbits; f has precisely two representations as a mating, which are of the form $f \cong R \amalg Q \simeq P \amalg R$.

When f is a Lattès map, different representations are known except in the case c) of $1/6 \amalg 1/6$. The Shishikura Algorithm can be used to identify the particular map f in the case of type $(2, 2, 2, 2)$, and we have only one quadratic map of type $(2, 4, 4)$. Combinatorial arguments show that there are basically nine, respectively three, matings of these types; see Sections 4 and 6 in [28].

- Case a) of type $(2, 2, 2, 2)$ is $1/4 \amalg 1/4 \simeq 23/28 \amalg 13/28 \simeq 13/28 \amalg 23/28 \simeq 53/60 \amalg 29/60 \simeq 29/60 \amalg 53/60$.
- Case b) of type $(2, 2, 2, 2)$ is given by $1/12 \amalg 5/12 \simeq -1/12 \amalg 5/12$.

- Case d) of type $(2, 2, 2, 2)$ is $1/6 \amalg 5/14 \simeq 5/14 \amalg 1/6 \simeq 3/14 \amalg 3/14 \simeq 3/14 \amalg 1/2 \simeq 1/2 \amalg 3/14 \simeq 5/6 \amalg 1/2 \simeq 1/2 \amalg 5/6$.
- Type $(2, 4, 4)$ is given by $\pm 1/4 \amalg 1/2 \simeq 5/12 \amalg \pm 1/6 \simeq 13/28 \amalg \pm 3/14$.

The following technique for producing shared matings is based on the representation of matings as repelling-preperiodic captures according to Theorem 2.8.

Proposition 3.4 (Shared matings from precaptures)

Suppose $P(z) = z^2 + p$ is geometrically finite, with $p \neq -2$, $p \neq 1/4$, and p not in the main cardioid. There are countably many pairs of preperiodic angles θ_1, θ_2 such that: the corresponding dynamic rays land together at a preperiodic pinching point $z_1 \in \partial\mathcal{K}_p$, which is not postcritical and not in the same branch at α_p as p , and the branch or branches of z_1 between these rays do not contain postcritical points of P or iterates of z_1 . Then we have $P \amalg Q_1 \simeq P \amalg Q_2$ with $q_i = \gamma_M(-\theta_i)$. Moreover, $P \amalg Q_1 \cong P \amalg Q_2$ if β_p is not between these rays.

Proof: We need to exclude $p = 1/4$ and the main cardioid, because \mathcal{K}_p would have no pinching points, and $p = -2$, because rays landing together at the interval \mathcal{K}_{-2} are never homotopic with respect to the postcritical set. If P is postcritically finite, Proposition 2.7 shows that the precaptures $\varphi_{\theta_1} \circ P$ and $\varphi_{\theta_2} \circ P$ are combinatorially equivalent. So the canonical obstructions and the essential maps are equivalent as well. According to the proof of Theorem 2.8, given in [27], the essential maps are equivalent to the geometric matings. By continuity according to Section 2.2, the result extends to geometrically finite P : ■

- The example $11/24 \amalg 13/56 \cong 11/24 \amalg 15/56$ enjoys the following property: the latter mating has an equator and a simple pseudo-equator, while the former does not have either.
- As another example, consider $p = \gamma_M(59/240)$ and $q = \gamma_M(63/240)$. Applying this construction to P and to Q gives $P \amalg P \cong P \amalg Q$ and $Q \amalg P \cong Q \amalg Q$. Here the first and second polynomials may be interchanged on both sides, so we have four representations of the same rational map; in particular there are shared self-matings $P \amalg P \cong Q \amalg Q$, and the flipped matings $P \amalg Q \cong Q \amalg P$.
- When P is the Basilica, all pinching points are preimages of α_p . Since none of these is iterated behind itself, shared matings are obtained from any pinching point z_1 , which is not α_p or behind it. Dudko [17] has shown that these are the only shared Basilica matings, since the parameter space is described as a mating of \mathcal{M} and \mathcal{K}_p . The simplest example is given by $P \amalg (z^2 \pm i)$: the geometric matings are distinct and complex conjugate, and both affine conjugate to $\frac{z^2+2}{z^2-1}$. The example $P \amalg 5/24 \simeq P \amalg 7/24$ is illustrated with a video of slow mating on www.mndynamics.com. Aspenberg [2] constructs the semi-conjugation from the Basilica to the rational map, beginning with the Boettcher map; in this alternative approach, shared matings are obtained from a non-unique labeling of Fatou components by bubble rays.
- Shared matings in the family of Chebychev maps are discussed in Section 3.3. In certain cases, lower bounds on the multiplicity are obtained from homotopic rays according to Proposition 3.4, or upper bounds are obtained directly.

- When z_1 is a branch point of \mathcal{K}_p , there may be more than two parameters q_i . In Theorem 3.8 of Section 3.4, unboundedly shared Airplane matings with small preperiods and periods are constructed. Although the Airplane does not contain any branch point, this is achieved by choosing q_i with a common branch point in \mathcal{K}_q .
- If f is a critically preperiodic rational map of degree $d \geq 2$ with three or four postcritical points, a pseudo-equator may produce several unmatings by choosing different pseudo-isotopies to its preimage [38]. A higher multiplicity is obtained when there are degenerate critical points, or when a critical point is mapped to another one. Probably the only quadratic example is the Lattès map of type (2, 4, 4). See [21] for related results on NET maps.

Remark 3.5 (Finite multiplicity)

If f is a postcritically finite quadratic rational map, can there be infinitely many representation as a mating $f \simeq P \amalg Q$?

- When f is hyperbolic, there are only finitely many candidates for P and Q , since there are only finitely many quadratic polynomials with a given superattracting period.
- When one critical point is periodic and one is preperiodic, finiteness is not obvious. For a specific family, finiteness is shown in Theorem 3.7 of the following section, using similar techniques as in the Lattès case.
- When both critical points are preperiodic, finiteness is shown for Lattès maps in [28]. Probably the techniques can be applied to a few other examples of small preperiod and period, but a general proof shall be harder.

3.3 Shared matings in the Chebychev family

Let us define a Chebychev map as a quadratic rational map of the form $f(z) = f_a(z) = \frac{z^2 - a - 2}{z^2 + a}$, $a \neq -1$, for which $f(\infty)$ is pre-fixed: $\infty \Rightarrow 1 \rightarrow -1 \uparrow$. This family contains matings with the Chebychev polynomial in particular:

Proposition 3.6 (Chebychev maps as matings)

Suppose $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ are geometrically finite and not in conjugate limbs of the Mandelbrot set \mathcal{M} . Then the geometric mating is affine conjugate to a Chebychev map, $f_a \simeq P \amalg Q$, if and only if P and Q are one of the following forms:

- Q is the Chebychev polynomial $Q(z) = z^2 - 2$ and p is not in the 1/2-limb of \mathcal{M} .
- p is in the k/r -limb of \mathcal{M} , and $q = \gamma_{\mathcal{M}}(-\theta)$, where θ is one of the r angles at $-\alpha_p \in \mathcal{K}_p$ (which depend only on k/r).
- For a rotation number $k/r \neq 1/2$, denote the angles of the k/r -wake by θ_{\pm} and let $\theta = (\theta_- + \theta_+)/2$ be the unique angle of preperiod 1 and period r in that limb. If $q = \gamma_{\mathcal{M}}(\theta)$, then P must be in the closed wake of the primitive hyperbolic component Ω with the root $\gamma_{\mathcal{M}}(-2\theta)$.

The Petersen transformation [39] maps symmetric rational maps to Chebychev maps, such that self-matings are mapped to Chebychev matings; see also Remark 4.4 in [28]. In the previous section the example of shared self-matings $59/240 \amalg 59/240 \cong$

63/240 \amalg 63/240 was obtained from Proposition 3.4; now the Petersen transformation gives the shared Chebychev mating 59/240 \amalg 1/2 \cong 63/240 \amalg 1/2.

Proof of Proposition 3.6: As explained in Figure 1, instead of saying that angles of $z \in \mathcal{K}_p$ and $w \in \mathcal{K}_q$ are complex conjugate, we may say that $z \in \mathcal{K}_p$ shares an angle with $\bar{w} \in \mathcal{K}_{\bar{q}}$, or connect $\mathcal{K}_{\bar{p}}$ to \mathcal{K}_q as well. In the formal mating $g = P \sqcup Q$, the ray-equivalence class of $g^2(\infty)$, corresponding to $\bar{Q}^2(0) = \bar{Q}(\bar{q})$, is fixed. By Proposition 2.6, it must contain a fixed point of P or \bar{Q} . If this is β_p or $\beta_{\bar{q}}$, the fixed class is the 0-ray and $\bar{Q}(\bar{q}) = \beta_{\bar{q}}$, which is case a).

b) Now suppose that $\bar{Q}(\bar{q})$ is in the same ray-equivalence class as α_p and $p \in \mathcal{M}_{k/r}$. Then the critical value \bar{q} is connected to $-\alpha_p$. This connection must be direct, since \mathcal{K}_p does not contain another pinching cycle of ray period r . So \bar{q} shares an external angle with $-\alpha_p$ and all of these angles may occur, since none is in the same sector at α_p as the critical value p , and q is not in the conjugate limb. The r angles belong to different Misiurewicz points in fact, since otherwise some $P \sqcup Q$ would have a closed ray connection.

c) Consider $q \in \mathcal{M}_{k/r}$ and \bar{P} such that $Q(q)$ is in the same ray-equivalence class as α_q . The points are not equal, because the preperiod would have to be $\geq r > 1$. So the ray connection must have length 2, since length ≥ 4 would require additional pinching cycles of ray period r in $\mathcal{M}_{k/r}$. Thus q has the external angle θ defined above, and $\mathcal{K}_{\bar{p}}$ must contain a pinching cycle of period and ray period r , which connects the cycle of $2\theta = \theta_- + \theta_+$ to that of θ_{\pm} . This cycle of $\mathcal{K}_{\bar{p}}$ persists from a primitive hyperbolic component $\bar{\Omega}$ before \bar{p} .

It remains to show that $\bar{\Omega}$ exists and is unique. In the dynamic plane of Q , the r rays landing at α_q define r sectors W_1, \dots, W_r with $0 \in W_r$ and $q \in W_1$, such that Q is a conformal map $W_1 \rightarrow W_2 \rightarrow \dots \rightarrow W_{r-1} \rightarrow W_r$ and the sectors are permuted with rotation number k/r . The external rays with angles $2^{i-1}\theta_{\pm}$ bound W_i for $1 \leq i \leq r$. Now W_i contains $2^{i-1}\theta$ as well for $2 \leq i \leq r-1$ and W_r has both $2^{r-1}\theta$ and $2^r\theta = -\theta$. For $r \geq 3$ it follows that 2θ has exact period r . We are looking for a primitive orbit portrait [41] connecting each angle in $\{2^i\theta \mid 1 \leq i \leq r\}$ to a unique angle in $\{2^i\theta_- \mid 1 \leq i \leq r\} = \{2^i\theta_+ \mid 1 \leq i \leq r\}$.

Starting in W_r , connect $2^{r-1}\theta$ to either $2^{r-1}\theta_-$ or to $2^{r-1}\theta_+$, such that $2^r\theta$ is not separated from the other angles. Pull the connection back until 2θ is connected to $2\theta_-$ or $2\theta_+$. The complement of the $r-1$ disjoint small sectors is connected, so we can connect the remaining angles $2^r\theta$ and θ_+ or θ_- as well. This construction gives a valid orbit portrait and defines $\bar{\Omega}$, which has the external angles 2θ and $2\theta_-$ or $2\theta_+$. Note that it is a narrow component, i.e., its angular width is $1/(2^r - 1)$ and there is no component of period $\leq r$ behind $\bar{\Omega}$. To show that $\bar{\Omega}$ is unique, suppose we had started by connecting $2^{r-1}\theta$ with an angle not bounding W_r and pulled it back. This pullback would follow the rotation number k/r as well and the small sectors would overlap, the leaves would be linked. \blacksquare

Case b) provides maps from limbs of \mathcal{M} to the Chebychev family, which are partially shared according to Proposition 3.4: e.g., for P geometrically finite in the 1/2-limb, consider the geometric matings corresponding to $P \amalg \pm 1/6$, i.e. $p \mapsto f_a \simeq P \amalg \pm 1/6$. These two maps agree on the small Mandelbrot set of period 2, but in general do not agree on its decorations. Likewise, for p in the 1/3-limb, we have three maps corresponding to $P \amalg 3/14$, $P \amalg 5/14$, and $P \amalg 13/14$, which agree on the small

Mandelbrot set of period 3. In the decorations, two of the maps may agree on certain veins, but in general the third one will be different: the relevant rays are no longer homotopic. Note that according to case c), some of these Chebychev maps are represented by $\tilde{P} \amalg 3/14$ as well, with \tilde{p} in the Airplane wake. In particular, we have $1/7 \amalg 3/14 \cong 1/7 \amalg 5/14 \simeq 1/7 \amalg 13/14 \simeq 3/7 \amalg 3/14$. Under the Petersen transformation mentioned above, this Chebychev map is the image of $1/7 \amalg 3/7 \simeq 3/7 \amalg 1/7$, which is a symmetric map but not a self-mating.

Theorem 3.7 (Chebychev maps as shared matings)

Matings $P \amalg Q$ in the Chebychev family with hyperbolic P have non-uniformly bounded multiplicity:

1. *Suppose $f = f_a$ is a Chebychev map, such that $z = 0$ is n -periodic. Then there are at most a finite number of representations $f_a \simeq P \amalg Q$.*
2. *For each rotation number k/r , there is a unique Chebychev map $f = f_a$, such that $z = 0$ is r -periodic and the fixed point $-1 = f^2(\infty)$ is a common boundary point of the r immediate basins, which are permuted with rotation number k/r . This map has precisely $r + 1$ realizations as a geometric mating, $f \simeq P \amalg Q$, when $r \geq 3$; for $r = 2$ there are only 2 representations.*

Proof: 1. P will be n -periodic, so there are only finitely many possibilities for P . We must see that r is bounded in cases b) and c). But in both cases we have $r \leq n$, since the wakes of period r are narrow: in case b) this is a basic property of limbs, and in case c) it was noted in the proof of Proposition 3.6.

2. In case a) of Proposition 3.6, $z = -1$ corresponds to the ray-equivalence class of angle 0, which does not touch a hyperbolic component of P . In cases b) and c), the rotation number at -1 is precisely k/r , so the value of k/r from the proposition must be the same as in the hypothesis of the theorem; case c) is excluded for $k/r = 1/2$. In both cases, there is only one hyperbolic component of period r in the limb or wake. It remains to show that f_a is unique, so that the $r + 1$ (or two) matings actually give the same map. Intuitively, this follows from the fact that the hyperbolic component of f_a bifurcates from the hyperbolic component where -1 is attracting; the multiplier map with $\rho = \frac{-4}{a+1}$ is injective for $|a + 1| \geq 4$. It can be proved by Thurston rigidity, since there is a forward-invariant graph connecting the postcritical points, which depends only on k/r up to isomorphy. So all possible maps f_a are combinatorially equivalent, affine conjugate, and equal. — Note that the case of $k/r = 2/5$ was discussed in the Introduction and in Figure 1. ■

3.4 Unboundedly shared Airplane matings

Denoting the Rabbit by R and the Airplane by A , we have seen in the previous Section 3.3 that $R \amalg 3/14 \cong R \amalg 5/14 \simeq R \amalg 13/14 \simeq A \amalg 3/14$. This example belongs both to the Chebychev family and to the family V_3 with a 3-periodic critical point. Unboundedly shared matings were obtained in Theorem 3.7.2 by increasing both the period of the hyperbolic polynomial P and the ray period of the Misiurewicz polynomial Q . Another example is obtained below, where Q is always the Airplane, and the preperiod of P is unbounded. The proof will be a simple application of Proposition 3.4 again. Airplane matings with unbounded multiplicity are due to Rees [46]

with hyperbolic polynomials P , such that the period of P grows exponentially with the multiplicity.

Theorem 3.8 (Unboundedly shared Airplane matings)

For the Airplane q and $n = 3, 5, 7, \dots$, there are n Misiurewicz parameters p_*, p_2, \dots, p_n such that the geometric matings agree, $f \cong P_i \sqcup Q$ for all $i = *, 2, \dots, n$. Here all p_i have preperiod $n + 1$, p_* has period 1 and p_2, \dots, p_n have period n ; so $f(\infty)$ has preperiod $n + 1$ and period 1. The statement remains true for large n , when q is any geometrically finite parameter behind $\gamma_M(5/12)$ and before the Airplane. E.g., q may be the Misiurewicz point $\gamma_M(41/96)$ as well.

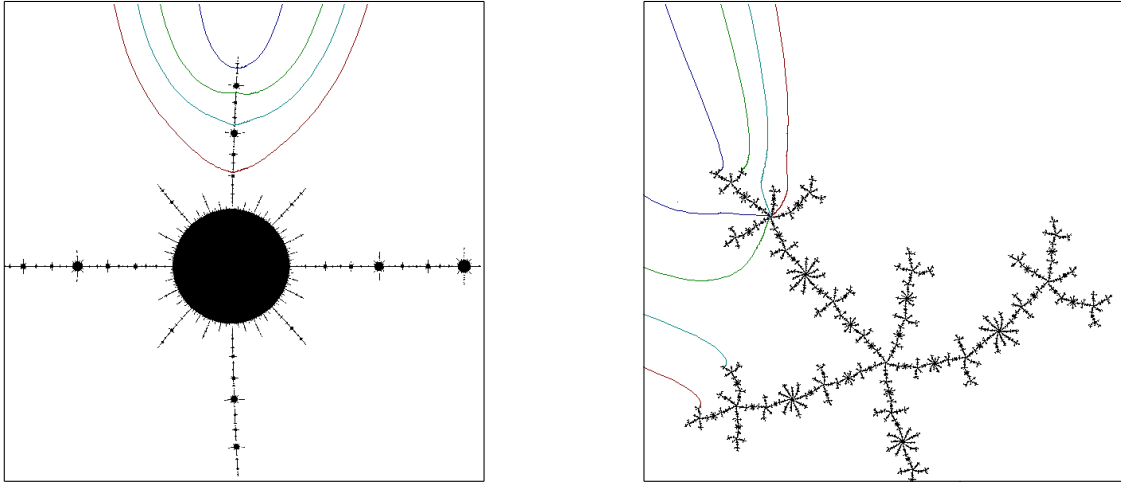


Figure 2: Consider the formal mating $g = P \sqcup Q$, with the Airplane \mathcal{K}_q shown rotated on the left, and \mathcal{K}_p on the right for some p in the 2/5-limb. According to the proof of Theorem 3.8, there are eight angles $\theta_2, \dots, \theta_5, \theta'_2, \dots, \theta'_5$, such that $-\theta_i$ and $-\theta'_i$ land together at the Airplane $\partial\mathcal{K}_q$, while θ_i land together at $\partial\mathcal{K}_p$. So the eight rays belong to a preperiodic ray-equivalence class of diameter four; actually there are two more rays crossing the Airplane on the real axis. Now there are five parameters $p = p_*, p_2, \dots, p_5$, such that this ray-equivalence class contains the critical value p , and it is shown that the corresponding matings define the same rational map f .

Proof: Denote the Airplane parameter by q and fix $n \in \{3, 5, 7, \dots\}$; let c be the first center of period n behind the Misiurewicz point $\gamma_M(5/12)$. The orbit of the characteristic point z_1 is ordered as

$$\begin{aligned} z_1 &< \gamma_c(5/12) < z_{n-1} < z_{n-3} < \dots < z_6 < z_4 < \alpha_c < \\ &< z_3 < z_5 < \dots < z_{n-2} < z_n < 0 < -\alpha_c < z_2 ; \end{aligned} \tag{1}$$

the critical orbit (z_i^*) is similar with $z_n^* = 0$. This ordering is well-known from discussions of Sharkovskii combinatorics. It can be checked with dynamic angles as follows: first, note that the order of the critical orbit is compatible with the assumption that $f_c : [z_1^*, 0] \rightarrow [z_1^*, z_2^*]$ is strictly decreasing and $f_c : [0, z_2^*] \rightarrow [z_1^*, z_3^*]$ is strictly increasing, so this defines a unique real polynomial. Let Θ_1 be the larger angle at z_1 and denote its iterates under doubling by Θ_i . Then

$$\begin{aligned} 0 &< \Theta_2 < 1/6 < \Theta_n < \Theta_{n-2} < \dots < \Theta_5 < \Theta_3 < 1/3 < \\ &< 1/2 < \Theta_1 < 7/12 < \Theta_{n-1} < \Theta_{n-3} < \dots < \Theta_6 < \Theta_4 < 2/3 < 1 , \end{aligned} \tag{2}$$

since the derivative of the real polynomial $f_c(z) = z^2 + c$ is negative for $z < 0$ and then f_c swaps the lower and upper half-planes. Reading off binary digits gives $\Theta_1 = .\overline{100101\dots 010}$, which is the largest n -periodic angle less than $7/12 = .\overline{100\overline{1}}$. Reversing these arguments, it follows that the center defined by $\gamma_M(\Theta_1)$ is real and the orbit is as given by (1). Each z_i has two external angles, Θ_i and $1 - \Theta_i$. Note that $f_c : [0, \beta_c] \rightarrow [c, \beta_c]$ is increasing; taking preimages of z_i with respect to this branch gives strictly preperiodic points except for z_3 , which has a periodic preimage z_2 on the positive half-axis.

Now consider any parameter p in the limb with rotation number k/n , $k = (n-1)/2$. The wake is bounded by $0 < \theta_- < \theta_+ < 1/3$. We have $\theta_+ = .\overline{0101\dots 010}$, since this is the largest n -periodic angle less than $1/3 = .\overline{0\overline{1}}$, or by sketching an n -Rabbit. So $\theta_+ = \Theta_3$ and $\theta_- = \Theta_5$; note that $\Theta_1 = 1/2 + \theta_+/4$ is an instance of the Douady Magic Formula from Proposition 2.1. — The critical value p of $f_p(z) = z^2 + p$ is in the sector at α_p bounded by the dynamic rays with the angles θ_{\pm} . This sector is mapped injectively for $n-1$ iterations; the image sector contains 0 , $-\alpha_p$, and a unique point in $f_p^{-1}(-\alpha_p)$. Thus the original sector contains unique preimages of α_p with preperiods n and $n+1$, respectively. Denote the angles of the latter by $\theta_1 < \dots < \theta_n$. Under n iterations, these are mapped to angles at $-\alpha_p$, such that θ_1 gives the smallest angle in $[1/2, 1]$ and θ_n gives the largest angle in $[0, 1/2]$. So under $n+1$ iterations, θ_1 is mapped to $\theta_+ = \Theta_3$ and θ_n is mapped to $\theta_- = \Theta_5$.

Next, let us look at the Airplane Julia set \mathcal{K}_q with $Q(z) = f_q(z) = z^2 + q$. As the parameter was shifted from c to q , the n -periodic points with angles Θ_i moved holomorphically; in particular the pre-characteristic points corresponding to $\pm z_n$ bound an interval containing the real slice of the Airplane Fatou component around 0 . Consider the Fatou component of f_c at z_3 ; it defines an interval in \mathcal{K}_q , which contains a unique preperiodic component Ω of preperiod $n-3$. Its largest antenna in the upper halfplane has angles in a subset of $[\Theta_5, \Theta_3] = [\theta_-, \theta_+]$. Since f_q^{n-3} maps it to the largest antenna on the upper side of the Fatou component around 0 , f_q^{n-2} maps it behind the component around q . Then it is behind the component around $f_q(q)$, then to the right of the component at 0 , and finally we see that f_q^{n+1} maps the antenna of Ω to the interval $(\gamma_q(4/7), \beta_q]$. Denote by x_i the preimage of the n -periodic point with angle Θ_i , then x_3 has preperiod n and the others have preperiod $n+1$. On the other hand, the angles θ_i are the only angles of preperiod $n+1$ in (θ_-, θ_+) that are iterated to some Θ_j . Recalling that θ_1 is iterated to $\theta_+ = \Theta_3$, we see that each θ_i with $i \neq 1$ lands at some x_j with $j \neq 3$. Denote the other angle by $\theta'_2, \dots, \theta'_n$; it is in (θ_-, θ_+) as well, since the antenna is contained in an open half-strip bounded by these rays and a real interval.

Finally, define the Misiurewicz parameters $p_* = \gamma_M(\theta_1) = \dots = \gamma_M(\theta_n)$ and $p_i = \gamma_M(\theta'_i)$, $i = 2, \dots, n$. Now p_* is of α -type by construction, so it has preperiod $n+1$ and period 1. The p_i are endpoints, since there is no other hyperbolic component of period n in the k/n -limb; they are pairwise different in particular. Note that for $i = 2, \dots, n$, the rays $\mathcal{R}_q(-\theta'_i)$ and $\mathcal{R}_q(-\theta_i)$ land together as well and the landing point never returns to this wake, so the two rays are homotopic with respect to its orbit and to the real orbit of q , and the precaptures are equivalent: by Proposition 3.4, the matings $Q \amalg P_i \cong Q \amalg P_*$ agree, as do $P_i \amalg Q \cong P_* \amalg Q$. — For the example of $k/n = 2/5$, Figure 2 shows the rays with angles $-\theta_i, -\theta'_i$ landing pairwise at $\partial\mathcal{K}_q$, and the rays with angles θ_i, θ'_i landing at $\partial\mathcal{K}_{p_*}$, at a preimage of α_{p_*} and at

endpoints, respectively.

The landing pattern at $\partial\mathcal{K}_q$ is stable for parameters q between c of period n as above and the Airplane, but the relevant antenna will bifurcate when q is too far behind the Airplane. ■

Note that we have constructed n different matings giving the same rational map, but in contrast to Theorem 3.7, no upper bound on the multiplicity is known in this case. — Assuming that the map $\mathcal{M}_{k/r} \rightarrow V_3$, $P \mapsto f \cong P \amalg Q$ is continuous, there will be self-intersections of the image corresponding to these shared matings.

3.5 Counterexamples to continuity of mating

Geometric mating is not jointly continuous on the subset of $\mathcal{M} \times \mathcal{M}$ where it can be defined. The first three examples below are due to Epstein [19, 9]. Note that all of these techniques involve neutral parameters, and that they do not exclude separate continuity. For specific one-dimensional slices with Q fixed, partial results on continuity have been obtained by Dudko [17] and by Ma Liangang [34].

— *Special thanks to Adam Epstein for explaining unpublished results.*

- Let f_λ be a quadratic polynomial with a fixed point of attracting multiplier λ . For $|\lambda| < 1$, $|\mu| < 1$ there are explicit rational maps $F_{\lambda,\mu} \simeq f_\lambda \amalg f_\mu$. Suppose $\lambda, \mu \rightarrow 1$ tangentially, such that the third multiplier ν is constant. Then if $F_{\lambda,\mu}$ converges to a quadratic rational map, it will depend on ν , so there are oscillating sequences as well. Note that convergence may depend on a normalization allowing the collision of the respective fixed points; in a different normalization, $F_{\lambda,\mu}$ might converge to a map of degree one or to a constant as well.
- Results on shared matings with cluster cycles by Sharland [52, 53] are reported in Section 3.2. For rotation number $1/n$, we have $f_n \cong R_n \amalg Q_n \simeq P_n \amalg R_n$, where the center parameters correspond to the following roots: $r_n \sim \gamma_M(1/(2^n-1)) = \gamma_M(2/(2^n-1))$, $q_n \sim \gamma_M(-3/(2^n-1)) = \gamma_M(-4/(2^n-1))$, and $p_n \sim \gamma_M((2^{n-1}-1)/(2^n-1)) = \gamma_M(2^{n-1}/(2^n-1))$. Then $r_n \rightarrow r_0 = 1/4 = \gamma_M(0)$, $q_n \rightarrow q_0 = 1/4 = \gamma_M(0)$, and $p_n \rightarrow p_0 = -2 = \gamma_M(1/2)$. Now if mating was continuous, we should have $R_0 \amalg Q_0 \simeq P_0 \amalg R_0$; both geometric matings exist, the former has two parabolic basins and the latter has one.
- For a parabolic or bounded-type Siegel parameter p on the boundary of the main cardioid with angle θ and the real parameter q defined by the Douady Magic Formula $\Theta = 1/2 + \theta/4$ according to Proposition 2.1, consider the geometric mating $f_\theta \cong P \amalg Q$, which exists according to Blé-Valdez [4, 5]. When θ is irrational, then $f_\theta^2(\infty) = 0$, since the corresponding point in \mathcal{K}_q has the angles $\pm 2\Theta = \pm\theta/2$ and the critical point of P has $\theta/2$ as well. But when θ is rational, then either 0 is in a parabolic basin and ∞ is preperiodic, or there are disjoint cycles of parabolic basins; in both cases $f_\theta^2(\infty) \neq 0$. So approximating a rational angle with irrational ones gives a contradiction to continuity.
- Theorem 3.9 below uses similar ideas to show that the limit is different from the expected one; since only rational angles are used, no special arguments are

needed to show matability. Here both p_n and q_n are Misiurewicz polynomials; a concrete example is given below as well.

- Shared matings according to Theorem 3.7 can be used to produce several counterexamples to continuity; here p_n is hyperbolic and q_n is Misiurewicz. Again, the contradiction comes either from a different number of parabolic Fatou cycles, or from an expected limit outside of the Chebychev family.
- Different kinds of discontinuity may be expected in higher degrees. E.g., with cubic polynomials $f_a(z) = z^3 + az^2$, the mating $f_a \amalg f_{-\bar{a}}$ gives an antipode-preserving rational map [6]. The former bifurcation locus shall be locally connected at parabolic parameters, while the latter is not. So for suitable sequences of postcritically finite polynomials, there will be an oscillatory behavior.

Theorem 3.9 (Discontinuity with bitransitive family)

Consider a sequence of rational angles $\theta_n \rightarrow \theta_0$, such that θ_n and $2\theta_n$ are preperiodic for $n \geq 1$, $2\theta_0$ is periodic, and θ_0 may be either unless θ_0 and $2\theta_0$ belong to the same root. Set $p_n = \gamma_M(\theta_n)$ and $q_n = \gamma_M(-2\theta_n)$ for $n \geq 0$. Then the sequence of geometric matings $f_n \cong P_n \amalg Q_n$ does not converge to $f_0 \cong P_0 \amalg Q_0$.

Proof: First, note that θ and 2θ are never in the same limb, unless both are angles of the root. Thus all geometric matings under consideration exist. Since the angle θ_n of $p_n \in \mathcal{K}_{p_n}$ is complex conjugate to an angle $-\theta_n$ of $0 \in \mathcal{K}_{q_n}$, there is a direct ray connection between these two points, and the rational map satisfies $f_n(0) = \infty$. We have $f_n \not\rightarrow f_0$ since $f_0(0) \neq \infty$: while $z = \infty$ has an infinite orbit converging to a parabolic cycle of f_0 , $z = 0$ either has a finite orbit or it converges to a different parabolic cycle. — This phenomenon seems to be analogous to parabolic implosion, if we are looking at the polynomials Q_n or at precaptures according to Proposition 2.7: $q_n = \gamma_{q_n}(-2\theta_n)$ converges to the critical value q_0 inside a parabolic Fatou component of Q_0 , but $\gamma_{q_0}(-2\theta_0)$ is a boundary point of this component. Of course, parabolic implosion looks different for the rational maps here, since the Julia set of f_n is all of $\widehat{\mathbb{C}}$. ■

A concrete example is given by $\theta_n = u_n/2^{2n}$ with $u_n = (2^{2n-1} + 1)/3$. Then p_n and q_n are β -type Misiurewicz points, converging to the Misiurewicz point $p_0 = i = \gamma_M(1/6)$ and the root $q_0 = -3/4 = \gamma_M(1/3)$, respectively, and the matings do not converge to the mating of the limits. Probably we have a parabolic 2-cycle in both cases, and Fatou components corresponding to a fat Basilica, but the limit of the matings has 0 and ∞ in different components of the Fatou set, while the mating of the limits has 0 in the Julia set at a preimage of the parabolic fixed point.

4 Short and long ray connections

We shall obtain explicit bounds on ray connections in Section 4.1, discuss special irrational ray connections in Section 4.2, search long ray connections algorithmically in Section 4.3, and give examples of cyclic ray connections in Section 5. The results provide partial answers to Questions 3.1–3.3, 3.5–3.7, and 3.9 in [9].

4.1 Bounding rational and irrational ray connections

When p is postcritically finite, every biaccessible point $z \in \partial\mathcal{K}_p$ will be iterated to an arc $[-\beta_p, \beta_p]$, then to $[\alpha_p, -\alpha_p]$, then to $[\alpha_p, p]$, and it stays within the Hubbard tree $T_p \subset \mathcal{K}_p$. In [42], Milnor discusses several aspects of the geometric and the topological mating $P \amalg Q$ with $p = q = \gamma_M(1/4)$. Every non-trivial ray connection will be iterated to a connection between points on the Hubbard trees, since every biaccessible point is iterated to the Hubbard tree T_p or $T_{\bar{q}}$. The two sides of the arcs of T_p are mapped in a certain way, described by a Markov graph with six vertices, such that only specific sequences of binary digits are possible for external angles of T_p . It turns out the only common angles of T_p and $T_{\bar{q}}$ are the 4-cycle of $3/15$ and some of its preimages. This fact implies that all ray connections between the Julia sets \mathcal{K}_p and $\mathcal{K}_{\bar{q}}$ are arcs or trees of diameter at most 3, so the topological mating exists by Proposition 2.5.

We shall consider an alternative argument, which is due to [55] in a cubic situation. It gives weaker results in the example of $1/4 \amalg 1/4$, but it is probably easier to apply to other cases: T_q is obtained by cutting away the open sector between the rays with angles $9/14$ and $11/14$, and its countable family of preimages, from \mathcal{K}_q . So no $z \in T_{\bar{q}}$ has an external angle in the open interval $(3/14, 5/14)$, or in its preimages $(3/28, 5/28)$ and $(17/28, 19/28)$. Now for every z on the arc $[\alpha_p, -\alpha_p]$, the angles on one side are forbidden. That shall mean that the corresponding rays do not connect z to a point in $T_{\bar{q}}$, but to an endpoint of $\mathcal{K}_{\bar{q}}$ or to a biaccessible point in a preimage of $T_{\bar{q}}$. This fact implies that every ray-equivalence class has diameter at most four, which is weaker than Milnor's result, but sufficient for the topological mating.

This argument shall be applied to another example, the mating of the Kokopelli P and the Airplane Q . Here $T_{\bar{q}} = T_q$ has no external angle in $(6/7, 1/7)$, and one side of $[\alpha_p, -\alpha_p]$ has external angles in $[1/14, 1/7]$. Treating preimages of α_p separately, it follows that no other point in \mathcal{K}_p is connected to two points in $T_{\bar{q}}$, and we shall see that all ray-equivalence classes are uniformly bounded trees. So the existence of the topological mating is obtained without employing the techniques of Theorem 2.3 by Thurston, Rees–Shishikura–Tan, and Rees–Shishikura. Moreover, this approach works for geometrically finite and infinite polynomials as well. E.g., q may be any real parameter before the Airplane root, and p be any parameter in the small Kokopelli Mandelbrot set. Note however, that only the topological mating is obtained here, not the geometric mating: there need not be a corresponding rational map.

To formulate the argument when \mathcal{K}_q is locally connected but Q is not postcritically finite, we shall employ a generalized Hubbard tree T_q : it is a compact, connected, full subset of \mathcal{K}_q , which is invariant under Q and contains an arc $[\alpha_q, q]$. If \mathcal{K}_q has empty interior and q is not an endpoint with irrational angle, there will be a minimal tree with these properties. When \mathcal{K}_q has non-empty interior, a forward-invariant topological tree need not exist, but we may add closed Fatou components to suitable arcs to define T_q . And when q is an irrational endpoint, we shall assume that it is renormalizable, and add complete small Julia sets to $T_{\bar{q}}$. — Note that in any case, every biaccessible point in \mathcal{K}_q will be absorbed by T_q , since $[\alpha_q, q] \subset T_q$.

Proposition 4.1 (Explicit bound on ray connections)

Consider ray-equivalence classes for the formal mating $g = P \sqcup Q$ of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, with \mathcal{K}_p and \mathcal{K}_q locally connected, and with a generalized Hubbard tree $T_q \subset \mathcal{K}_q$ as defined above. Now suppose that there is an open set of angles, such that no external angle of $T_{\bar{q}}$ is iterated to this forbidden set, and such that for an arc $[\alpha_p, -\alpha_p] \subset \mathcal{K}_p$, the external angles on one side are forbidden. Then:

1. Any point in \mathcal{K}_p has at most one ray connecting it to a point in the generalized Hubbard tree $T_{\bar{q}}$ of \bar{Q} .
2. All ray-equivalence classes have diameter bounded by eight, since each class is iterated to a tree of diameter at most four.
3. Moreover, there are no cyclic ray connections, so the topological mating $P \sqcup Q$ exists according to Proposition 2.5.

Proof: 1. By assumption, α_p has at least one forbidden angle, but there may be several allowed angles. Since these are permuted transitively by iteration, none of them is connected to $T_{\bar{q}}$. In particular, there is no ray connecting α_p to $\alpha_{\bar{q}}$, so p and q are not in conjugate limbs. Suppose $z \in \partial\mathcal{K}_p$ is not a preimage of α_p . If it had two rays connecting it to points in $T_{\bar{q}}$, this connection could be iterated homeomorphically until both rays are on different sides of the arc $(\alpha_p, -\alpha_p)$, contradicting the hypothesis since $T_{\bar{q}}$ is forward-invariant. (Even if z is precritical and reaches 0 with both rays on one side, the next iteration will be injective.)

2. Suppose C is any bounded connected subset of a ray-equivalence class. Iterate it forward (maybe not homeomorphically) until all of its preperiodic points have become periodic, all critical and precritical points have become postcritical, and all biaccessible points of $\mathcal{K}_{\bar{q}}$ have been mapped into $T_{\bar{q}}$. So C is a preimage of an eventual configuration C_∞ , which is a subset of a ray-equivalence class of diameter at most four, since it contains at most one biaccessible point of $T_{\bar{q}}$. E.g., it might be a periodic branch point of \mathcal{K}_p connected to several endpoints of $\mathcal{K}_{\bar{q}}$, or a point of $T_{\bar{q}}$ connected to two or more biaccessible points of \mathcal{K}_p , which are connected to endpoints of $\mathcal{K}_{\bar{q}}$ on the other side. In general, taking preimages will give two disjoint sets of the same diameter in each step, unless there is a critical value involved.

Now C_∞ contains at most one postcritical point of $\mathcal{K}_{\bar{q}}$. If there are several postcritical points of \mathcal{K}_p , then C_∞ is periodic, and preperiodic preimages contain at most one postcritical point of P . So when pulling back C_∞ , the diameter is increased at most twice, and it becomes at most 16. Actually, when C_∞ has diameter 4, neither postcritical point can be an endpoint of C_∞ , and some sketch shows that the diameter will become at most 8.

3. If C is a cyclic ray connection, it will be iterated to a subset of a tree C_∞ according to item 2. This means that in the same step, both critical points are connected in a loop C' , and $C'' = g(C)$ is a simple arc connecting the critical values $p \in \mathcal{K}_p$ and $\bar{q} \in \mathcal{K}_{\bar{q}}$. This cannot be a single ray, since p and q are not in conjugate limbs. Suppose that C'' is of the form $p - \bar{q}' - p' - \bar{q}$ with $\bar{q}' \notin T_{\bar{q}}$. Now \bar{q}' is biaccessible, so it will be iterated to $T_{\bar{q}}$, and then it must coincide with an iterate of \bar{q} by item 1. So C'' is not iterated homeomorphically, and p' must be critical or precritical. But then C'' would be contained in a finite periodic ray-equivalence class, and the critical value of P would be periodic, contradicting $p \in \partial\mathcal{K}_p$. The same arguments work to exclude longer ray connections between the critical values p and \bar{q} . ■

The following theorem provides large classes of examples. The parameter p is described by a kind of sector, and q is located on some dyadic or non-dyadic vein. More generally, q may belong to a primitive or satellite small Mandelbrot set, whose spine belongs to that vein. Let us say that q is centered on the vein:

Theorem 4.2 (Examples of matings with bounded ray connections)

When p and q are chosen as follows, with locally connected Julia sets, the topological mating $P \amalg Q$ exists according to Proposition 4.1:

- a) The parameter q is in the Airplane component or centered on the real axis before the Airplane component, and p in the limb M_t with rotation number $0 < t \leq 1/3$ or $2/3 \leq t < 1$.
- b) q is centered on the non-dyadic vein to $i = \gamma_M(1/6)$, and $p \in \mathcal{M}_t$ with rotation number $0 < t < 1/2$ or $2/3 < t < 1$.
- c) q is centered on the dyadic vein to $\gamma_M(1/4)$, and p is located between the non-dyadic veins to $\gamma_M(3/14)$ and $\gamma_M(5/14)$. This means $p \in \mathcal{M}_t$ with $1/3 < t < 1/2$, or $p \in \mathcal{M}_{1/3}$ on the vein to $\gamma_M(3/14)$ or to the left of it, or $p \in \mathcal{M}_{1/2}$ on the vein to $\gamma_M(5/14)$ or to the right of it. In particular, p may be on the vein to $\gamma_M(1/4)$, too.

Proof: The case of q in the main cardioid is neglected, because all ray connections are trivial. We shall consider the angles of $\mathcal{K}_{\bar{q}}$ according to Figure 1). When Q has a topologically finite Hubbard tree T_q , maximal forbidden intervals of angles are found by noting that orbits entering T_q must pass through $-T_q$. See, e.g., Section 3.4 in [26]. Denote the characteristic angles of the limb \mathcal{M}_t by $0 < \theta_- < \theta_+ < 1$. For $p \in \mathcal{M}_t$, the arc $[\alpha_p, \beta_p]$ has angles θ with $0 \leq \theta \leq \theta_+/2$ on the upper side and with $(\theta_- + 1)/2 \leq \theta \leq 1$ on the lower side.

- a) If q is in the Airplane component or before it, the Hubbard tree is the real interval $T_q = [q, q^2 + q]$. If q belongs to a small Mandelbrot set centered before the Airplane, T_q may contain all small Julia sets meeting an arc from q to $f_q(q)$ within \mathcal{K}_q . Now no $z \in T_{\bar{q}}$ has an angle in $(6/7, 1/7)$. So Theorem 4.2 applies when $\theta_+/2 < 1/7$ or $(\theta_- + 1)/2 > 6/7$. The strict inequality is not satisfied for $t = 1/3$ and $t = 2/3$. Then α_p and its preimages may be connected to three points in the Hubbard tree of the Airplane, but the diameter is bounded by four as well. Note that behind case a), with $q = \gamma_M(28/63)$ and $p = \gamma_M(13/63)$, there is a ray connection of length six.
- b) When q is centered on the vein to $\gamma_M(1/6)$, the interval $(11/14, 1/14)$ is forbidden, so $(13/14, 3/14)$ is forbidden for $T_{\bar{q}}$. We need $\theta_+/2 < 3/14$ or $(\theta_- + 1)/2 > 13/14$.
- c) For parameters q centered on the vein to $\gamma_M(1/4)$, the interval $(9/14, 11/14)$ is forbidden, so $(3/14, 5/14)$ is forbidden for $T_{\bar{q}}$. We shall take its preimage $(3/28, 5/28) \cup (17/28, 19/28)$ instead. When p is between the veins to $\gamma_M(3/14)$ and $\gamma_M(5/14)$, these two intervals are overlapping in a sense: every $z \in (\alpha_p, -\alpha_p)$ has all angles on one side in a forbidden interval. But then we have $p \in \mathcal{M}_t$ with $\theta_+/2 < 5/28$ or $(\theta_- + 1)/2 > 17/28$, so the forbidden intervals extend to $\pm\alpha_p$. ■

Example 4.3 (Bounded unlimited ray-equivalence classes)

Suppose q is chosen according to item a) or b), and p is constructed as follows. Take a primitive maximal component in the $1/3$ -limb, then a primitive maximal component in its $1/4$ -sublimb, a primitive maximal component in its $1/5$ -sublimb \dots , then the limit p has an infinite angled internal address with unbounded denominators. \mathcal{K}_p

is locally connected by the Yoccoz Theorem [24, 40], the topological mating exists according to Theorem 4.2, and there are branch points with any number of branches. So ray-equivalence classes are bounded uniformly in diameter, but not in size in the sense of cardinality.

4.2 More on irrational ray connections

If two parameter rays with angles $\theta_- < \theta_+$ accumulate at the same fiber of \mathcal{M} , it will intersect some dyadic vein in one point c , which is called combinatorially biaccessible. \mathcal{K}_c is locally connected and the dynamic rays with angles θ_{\pm} land at the critical value c , unless c is parabolic. See the references in Section 4.4 of [26]. The following proposition shows that cyclic ray connections for matings of biaccessible parameters can exist only in special situations, since they cannot be preserved for postcritically finite parameters behind them, where they are ruled out by Theorem 2.3 of Rees–Shishikura–Tan. Compared to Proposition 4.1, the situation is more general and the conclusion is weaker.

Proposition 4.4 (Cyclic irrational ray connections)

Consider the formal mating g of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, with parameters p and q not in conjugate limbs of \mathcal{M} .

- a) If p is geometrically finite and q is combinatorially biaccessible, or vice versa, or both are geometrically finite, then g does not have a cyclic ray connection.
- b) If both p and q are combinatorially biaccessible and not geometrically finite, then g has a cyclic ray connection, if and only if there is a ray connection between the critical values p and \bar{q} .

Proof: If both parameters are postcritically finite, the topological mating exists according to Theorem 2.3, and there can be no cyclic ray connection by the Moore Theorem. For hyperbolic or parabolic parameters, the ray connections will be the same as for the corresponding centers. In general, a ray connection between the critical values will have a cyclic preimage, so this connection does not exist in case a). Conversely, a cyclic connection C that does not contain precritical points of the same generation, will give a contradiction for postcritically finite parameters behind the current ones: it may be iterated, possibly non-homeomorphically, to a cyclic connection C_{∞} between points on the Hubbard trees, which are not critical or precritical, and this connection C_{∞} would survive. To see this for P , denote the external angles of the critical value p by $\theta_- < \theta_+$. Then no ray of C_{∞} will have an angle in $(\theta_-/2, \theta_+/2) \cup ((\theta_- + 1)/2, (\theta_+ + 1)/2)$. For parameters c behind p , the critical point is located in a strip bounded by these four rays, so no precritical leaf can separate the rays biaccessing points of \mathcal{K}_p in C_{∞} . (I have learned this technique from Tan Lei.) The same argument applies to \bar{q} and parameters behind it. ■

The following proposition is motivated by Question 3.7 in [9]. It deals with angles θ that are rich in base 2: the binary expansion contains all finite blocks, or equivalently, the orbit of θ under doubling is dense in \mathbb{R}/\mathbb{Z} . Angles with this property are rarely discussed for quadratic dynamics, but they form a subset of full measure in fact.

Proposition 4.5 (Rich angles and irrational ray connections)

Suppose the angle θ is rich in base 2. Set $\theta_n = 2^n\theta$ and $c_n = \gamma_{\mathcal{M}}(\theta_n)$ for $n \geq 1$.

Then c_n is a non-renormalizable endpoint of \mathcal{M} with trivial fiber, \mathcal{K}_{c_n} is a dendrite, and the critical orbit is dense in \mathcal{K}_{c_n} .

1. For $n \neq m$ consider the formal mating g of P and Q , with $p = c_n$ and $q = \overline{c_m}$. Then g has a ray-equivalence class involving the angle θ , which is an arc of length four. (Note that n and m may be chosen such that p and q are not in conjugate limbs, but it is unknown whether the topological or geometric mating exists.)
2. Let $\mathcal{X}_\theta \subset \mathcal{M}$ contain all parameters c , such that θ is biaccessing \mathcal{K}_c . Then \mathcal{X}_θ is totally disconnected, and it contains $c = -2$ and all c_n . So it has infinitely many point components, and it is dense in $\partial\mathcal{M}$.

Proof: Renormalizable and biaccessible parameters do not have dense critical orbits. The orbit of an angle at the main cardioid is confined to a half-circle [11]. By the Yoccoz Theorem [24, 40], \mathcal{K}_{c_n} is locally connected with empty interior.

1. Assuming $n < m$, pull back the ray of angle θ_m connecting postcritical points of \mathcal{K}_p and $\mathcal{K}_{\bar{q}}$. This ray connects two endpoints, so it forms a trivial ray-equivalence class. Since both points are postcritical of different generations, the diameter is doubled twice under iterated pullback (whenever there are two preimages, choose the component containing an image of θ).

2. For $c = -2$, every irrational angle is biaccessing, and for c_n , θ belongs to a critical or precritical point. By excluding all other cases, \mathcal{X}_θ can contain only these and maybe other non-renormalizable, postcritically infinite endpoints outside of the closed main cardioid, thus it has only point components. So suppose that θ is biaccessing \mathcal{K}_c :

For a Siegel or Cremer polynomial of period 1, at most precritical points or preimages of α_c are biaccessible [49], and the orbit of angles is not dense.

Pure satellite renormalizable parameters have only rational biaccessing angles outside of the small Julia sets.

When c is primitive renormalizable, the biaccessible points outside of the small Julia sets are iterated to a set moving holomorphically with the parameter, see Section 4.1 in [26]. It is contained in a generalized Hubbard tree T_c in the sense of Proposition 4.1.

When c is postcritically finite or biaccessible, all biaccessible points are absorbed by a topologically finite tree T_c . So their orbits are not dense in K_c unless $T_c = \mathcal{K}_c$, which happens only for $c = -2$.

It remains to show that \mathcal{X}_θ is dense in $\partial\mathcal{M}$: from a normality argument it is known that β -type Misiurewicz points are dense. For any Misiurewicz point $a = \gamma_M(\tilde{\theta})$ there is a subsequence with $\theta'_n \rightarrow \tilde{\theta}$. Then $c'_n \rightarrow a$, since Misiurewicz points have trivial fibers [47]. ■

4.3 Searching long ray connections

Consider rational ray-equivalence classes for the formal mating $g = P \sqcup Q$ with parameters p, q in non-conjugate limbs of \mathcal{M} . A non-trivial periodic ray connection requires pinching points in \mathcal{K}_p and \mathcal{K}_q with specific angles, which exist if and only if the parameters p, q are at or behind certain primitive roots or satellite roots. So a longer ray connection means that there are several relevant roots before the current parameters, and on the same long vein in particular. Let us say that a

ray connection is **maximal**, if it is not part of a longer connection existing for parameters behind the current ones. The following ideas were used to determine all maximal ray connections algorithmically for ray periods up to 24; see Table 1.

Per.	length 5	length 6	length 7	length 8	length 10	length 12
10	32 + 0	14 + 88	—	0 + 2	—	—
11	76 + 0	20 + 0	—	—	—	—
12	46 + 0	24 + 264	—	—	—	—
13	226 + 0	72 + 0	2 + 0	2 + 0	—	—
14	285 + 0	102 + 484	4 + 0	0 + 14	0 + 2	—
15	540 + 0	192 + 184	—	—	—	—
16	958 + 0	338 + 1060	4 + 0	2 + 10	0 + 4	—
17	1872 + 0	584 + 0	14 + 0	2 + 0	—	—
18	2814 + 0	884 + 2672	22 + 0	6 + 24	0 + 8	—
19	5856 + 0	1650 + 0	26 + 0	6 + 0	—	—
20	9534 + 0	2890 + 5244	58 + 0	4 + 42	0 + 8	—
21	16978 + 0	4900 + 898	64 + 0	4 + 0	—	—
22	30180 + 0	8423 + 10928	126 + 0	18 + 132	0 + 20	0 + 2
23	55676 + 0	15300 + 0	172 + 0	18 + 0	—	—
24	95830 + 0	25968 + 25312	242 + 0	24 + 96	0 + 28	—

Table 1: The length of maximal periodic ray connections depending on the ray period. The first number counts unordered pairs of periodic parameters with primitive-only connections, the second number is the connections including a satellite cycle. Length ≤ 4 is ubiquitous, length 5 appears already for periods 7 and 9, while length 6 happens for periods 4 and 6–9 as well. Length 9 and 11 was not found for periods ≤ 24 .

- Suppose $\mathcal{R}(\theta_1) \text{--} z_p \text{--} \mathcal{R}(\theta_2) \text{--} z_q \text{--} \mathcal{R}(\theta_3)$ is a step in the ray connection, then θ_1 and θ_2 belong to a cycle of angle pairs for \mathcal{K}_p , so there is a root before p with characteristic angles iterated to θ_1 and θ_2 . Likewise, there is a root before \bar{q} , whose characteristic angles are iterated to θ_2 and θ_3 . Conversely, given the angles θ_{\pm} of a root before p , we may determine conjugate angles for iterates of θ_+ under doubling, and check whether the root given by an angle pair is before \bar{q} ; it is discarded otherwise. So we record only the angle pairs of roots, and forget about the number of iterations and about which class in a cycle contains which characteristic point. Note that there is an effective algorithm to determine conjugate angles [8, 29], probably due to Thurston.
- A maximal ray connection should be labeled by highest relevant roots on the respective veins. However, a brute-force search starting with these roots will be impractical: varying both p and \bar{q} independently is too slow, and searching \bar{q} depending on p requires to match different combinatorics on two sides, since the characteristic point z_p corresponding to the highest root may be anywhere in the ray-equivalence class. So the idea is to run over all roots p_1 , try to build

a maximal ray connection on one side of the corresponding characteristic point, and to quit if the connection can be continued on the other side of that point.

- When a pinching point of satellite type is reached under the recursive application of the conjugate angle algorithm, we may double the length and stop. Alternatively, two separate algorithms may be used, one finding primitive-only ray connections starting from the first pinching point, and another one starting with the satellite-type point in the middle of the periodic ray-equivalence class.

For period 22, this algorithm has recovered the example given to Adam Epstein by Stuart Price [9]: for p behind $\{1955623/4194303, 1955624/4194303\}$ and q behind $\{882259/4194303, 882276/4194303\}$ there is a periodic ray-equivalence class of diameter 12. For 1/2-satellites only, the same algorithm was used for periods up to 40 in addition; this produced another example of diameter 14 for period 32, with p behind $\{918089177/4294967295, 918089186/4294967295\}$ and q behind $\{1998920775/4294967295, 1998920776/4294967295\}$. Note that, e.g., taking p and q as the corresponding centers, the formal mating will have non-postcritical long ray connections and the geometric mating shows clustering of Fatou components. For suitable preperiodic parameters behind these roots, the formal mating has long periodic ray-equivalence classes with postcritical points from both orbits, and preperiodic classes may have twice or up to four times the diameter of the periodic classes. — There are several open questions on long ray connections:

- What are possible relations between the linear order of roots on the veins to p and q , and the order of pinching points within a ray-equivalence class?
- For the lowest period with a particular diameter of a ray-equivalence class, is there always a 1/2-satellite involved?
- Is there a whole sequence with similar combinatorics and increasing diameters? If it converges, does the limit show non-uniformly bounded ray connections? Does the geometric mating of the limits exist? If not, does it have infinite irrational ray connections?
- Are there only short ray connections for self-matings and for matings between dyadic veins of small denominator, even though the Hausdorff dimension of biaccessing angles is relatively high according to [18]?

5 Cyclic ray connections

First we shall construct cyclic ray connections for the formal mating g of the Airplane $P(z) = z^2 - 1.754877666$ and the Basilica $Q(z) = z^2 - 1$. See Figure 3. All biaccessing rays of Q are iterated to the angles $1/3$ and $2/3$ at $\alpha_q = \alpha_{\bar{q}}$. Denote by C_0 the cyclic connection formed by the rays with angles $5/12$ and $7/12$. Pulling it back along the critical orbit of the Airplane gives nested cycles C_n around the critical value p , since g^3 is proper of degree 2 from the interior of C_1 to the interior of C_0 . Now C_n has 2^n points of intersection with \mathcal{K}_p , so its length is not uniformly bounded as

$n \rightarrow \infty$. Moreover, C_n connects points x_n converging to $x_\infty = \gamma_p(3/7) = \gamma_p(4/7)$ to points x'_n converging to $x'_\infty = \gamma_p(25/56) = \gamma_p(31/56)$. But these four rays are landing at endpoints of the Basilica, so the landing points $x_\infty \neq x'_\infty$ on the Airplane critical value component are not in the same ray-equivalence class. Thus the ray-equivalence relation is not closed. In fact, the limit set of C_n contains the boundary of the Fatou component, which meets uncountably many ray-equivalence classes. I am not sure what the smallest closed equivalence relation, or the corresponding largest Hausdorff space, will look like: it shall be some non-spherical quotient of the Basilica, with a countable family of simple spheres attached at unique points. This Hausdorff obstruction has been obtained independently by Bartholdi–Dudko [private communication]. — More generally, we have:

Theorem 5.1 (Unbounded cyclic ray connections)

Suppose p is primitive renormalizable of period m and \mathcal{K}_p is locally connected. Then there are parameters $c_ \prec c_0 \prec p$, such that for all parameters q with \bar{q} on the open arc from c_* to c_0 , the formal mating $g = P \sqcup Q$ has non-uniformly bounded cyclic ray connections. Moreover, these are nested such that the ray-equivalence relation is not closed. So the topological mating $P \amalg Q$ is not defined on a Hausdorff space.*

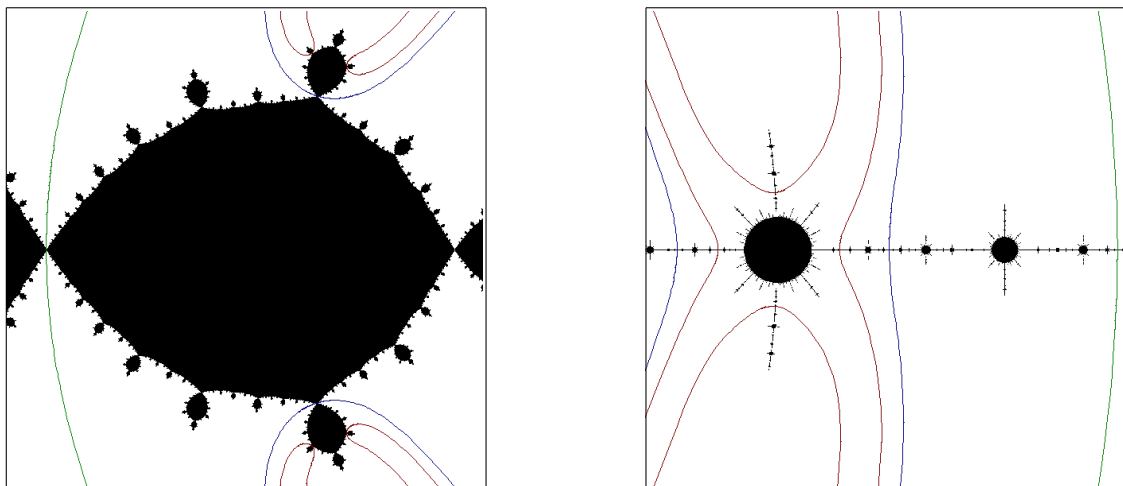


Figure 3: The formal mating g of the Airplane \mathcal{K}_p (on the right) and the Basilica \mathcal{K}_q (shown rotated on the left). The green ray connection C_0 has the angles $5/12$ and $7/12$. Suitable preimages C_1 (blue), C_2 (red), \dots form nested cycles around the critical value component of the Airplane. The nested domains are typical of primitive renormalization. The canonical obstruction of g is discussed in Figure 2 of [27].

Proof: In the dynamic plane of \mathcal{K}_p , denote the small Julia set around the critical value p by \mathcal{K}_p^m . There are preperiodic pinching points with $\alpha_p \preceq x_* \prec x_0 \prec x_1 \prec \mathcal{K}_p^m \prec x'_1$, such that P^m is a 2-to-1 map from the strip between x_1 and x'_1 to the wake of x_0 . Restricting these sets by equipotential lines in addition, we obtain a polynomial-like map, which is a renormalization of P . If the pinching points are branch points, the bounding rays must be chosen appropriately. We assume that x_1 and x'_1 are iterated to x_0 but never behind it, and x_0 is iterated to x_* but never behind it. More generally, x_* may be a periodic point. The construction of these points is well-known from primitive renormalization; see [47, 57, 29].

Since the points x_* and x_0 are characteristic in \mathcal{K}_p , there are corresponding Misiurewicz points c_* and c_0 in \mathcal{M} . (If x_* is periodic, then c_* is a root.) When the parameter \bar{q} is in the wake of c_* , or in the appropriate subwake, then x_0 will be moving holomorphically with the parameter and keep its external angles. When \bar{q} is chosen on the regulated arc from c_* to c_0 , then $\mathcal{K}_{\bar{q}}$ will be locally connected. In $\mathcal{K}_{\bar{q}}$ the point corresponding to x_0 has the same external angles as in \mathcal{K}_p , and no postcritical point is at this point or behind it. Thus the four rays defining the strip between $x_1, x'_1 \in \mathcal{K}_p$ are landing in a different pattern at $\mathcal{K}_{\bar{q}}$.

Now consider the formal mating $g = P \sqcup Q$. We shall keep the notation $x_i, p, \mathcal{K}_p, \bar{q}, \mathcal{K}_{\bar{q}}$ for the corresponding points and sets on the sphere. Since the two rays bounding the wake of x_0 , or the relevant subwake, are landing together at $\mathcal{K}_{\bar{q}}$, they form a closed ray connection C_0 . Its preimage is a single curve consisting of four rays, two pinching points in \mathcal{K}_p , and two pinching points in $\mathcal{K}_{\bar{q}}$. This can be seen on the sphere, since C_0 is separating the critical values of g , or in the dynamic plane of \bar{q} , since \bar{q} is not behind the point corresponding to x_0 . Now the new curve is pulled back with g^{m-1} to obtain C_1 , which is a closed curve connecting x_1 and x'_1 to two pinching points in $\mathcal{K}_{\bar{q}}$. By construction, g^m is proper of degree 2 from the interior of C_1 to the interior of C_0 , and the former is compactly contained in the latter. g^m behaves as a quadratic-like map around \mathcal{K}_p^m , but only points below the equator will converge to the small Julia set under iterated pullback.

Define the curves C_n inductively; they form strictly nested closed curves and the number of rays is doubled in each step. E.g., C_2 is intersecting \mathcal{K}_p in four points. The two preimages x_2 and x'_2 of x_1 are located between x_1 and x'_1 , while the two preimages of x'_1 belong to decorations of \mathcal{K}_p^m attached at the points with renormalized angles $1/4$ and $3/4$. We have $x_0 \prec x_1 \prec x_2 \prec \dots \prec \mathcal{K}_p \prec \dots \prec x'_2 \prec x'_1$. The limits x_∞ and x'_∞ are the small β -fixed point of \mathcal{K}_p^m and its preimage, the small $-\beta$. Now x_n and x'_n are connected by C_n , but x_∞ and x'_∞ are not ray-equivalent, because the former is periodic and the latter is preperiodic. ■

More generally, \bar{q} may be any parameter in the strip between c_* and c_0 , as long as its critical orbit does not meet the point corresponding to x_0 or get behind it. — Note that by taking iterated preimages of a finite ray-equivalence tree, you will merely get uniformly bounded trees: the diameter can be increased only when a critical value is pulled back to a critical point, which can happen at most twice according to Proposition 2.6: a finite irrational tree cannot be periodic, so it does not contain more than one postcritical point from each polynomial.

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The program Mandel provides several interactive features related to the Thurston Algorithm. It is available from www.mndynamics.com. A console-based implementation of slow mating is distributed with the [preprint](#) of [27].