

Ray connections and shared matings

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Abstract

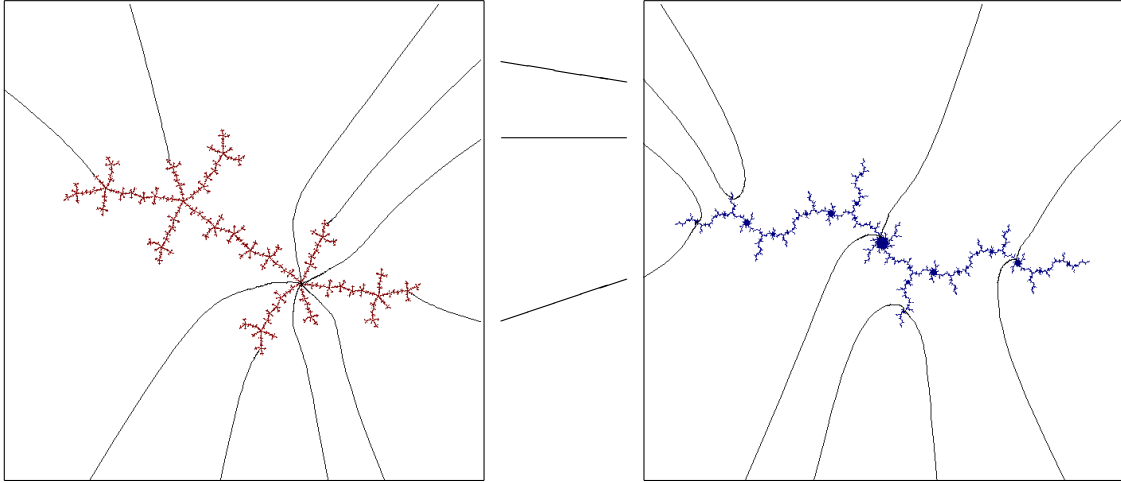
There is an alternative construction of mating, when at least one polynomial is preperiodic: shift the infinite critical value of the other polynomial to a preperiodic point. Taking homotopic rays, it gives simple examples of shared matings. Sequences with unbounded multiplicity of sharing, and slowly growing preperiod and period, are obtained both in the Chebychev family and for Airplane matings. Using preperiodic polynomials with identifications between the two critical orbits, an example of mating discontinuity is described as well.

1 Introduction

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Simple examples of shared matings and of mating discontinuity are obtained in **Section 3**. The rational map f above belongs to the same one-parameter family as matings with the Chebychev polynomial, but it is not of this form. There are five other representations as a mating: take the Rabbit with rotation number $2/5$ for P and suitable preperiodic parameters q_1, \dots, q_5 for Q , which are related to the angles at $-\alpha_p$. More generally, we have $P \amalg Q_i = P \amalg Q_j$ for all p in the small satellite Mandelbrot set, since the rays at $-\alpha_p$ are homotopic with respect to the postcritical set and so the precaptures are combinatorially equivalent. Taking higher rotation numbers gives shared matings with larger multiplicity. While it is obvious that a hyperbolic rational map has only a finite number of representations as a mating, this is not known in general when one or both of the critical points are preperiodic. Finiteness is shown here for Chebychev maps with one critical point periodic, and in [?] for Lattès maps. Examples with arbitrarily high multiplicity are obtained as well for matings of the Airplane with preperiodic polynomials; here preperiod and period are of the same order as the multiplicity, in contrast to the hyperbolic examples by Rees [?], where the period grows exponentially. — Simple ray connections can be used to define preperiodic matings with $f(0) = \infty$. This property is lost when preperiodic parameters converge to a parabolic parameter, confirming that mating is not jointly continuous. The mechanism is similar to geometrically infinite examples by Blé–Valdez–Epstein [?, ?], but here all maps are geometrically finite and matability does not require special arguments.

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Fchsh

Figure 1: A formal mating $g = P \sqcup Q$. The Julia set \mathcal{K}_p for the five-periodic center p corresponding to $\gamma_M(11/31)$ is shown on the right; the Misiurewicz Julia set \mathcal{K}_q with $q = \gamma_M(19/62)$ in the left image is rotated. (This does not change the set itself, but its external rays.) The ray connections and dynamics are discussed in the main text.

There are various ways to visualize the sets $\varphi_0(\mathcal{K}_p), \varphi_\infty(\mathcal{K}_q) \subset \widehat{\mathbb{C}}$ in the plane \mathbb{C} : instead of \mathcal{K}_q coming from ∞ , we may rotate the sphere such that \mathcal{K}_q is translated above or below \mathcal{K}_p , or to save space here, translated to the left or right and rotated.

In any case, $\varphi_0(\mathcal{R}_p(\theta))$ is connected with $\varphi_\infty(\mathcal{R}_q(-\theta))$; three connections are indicated between the two images. When discussing the combinatorics of a ray-equivalence class, we may avoid conjugation of several angles by assuming that $\mathcal{R}_p(\theta)$ connects to $\mathcal{R}_{\bar{q}}(\theta)$, but to draw these rays without crossing, you would need to stack two sheets of paper.

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2 Background

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2.1 Polynomials

2p

2.2 Matings

2m

2.3 Ray connections

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3 Mating as a map between parameter spaces

3

Mating provides a partial map from $\mathcal{M} \times \mathcal{M}$ to the moduli space of quadratic rational maps. This map is neither surjective, injective, nor continuous. The characterization of matings in terms of equators and pseudo-equators by Thurston–Wittner and Meyer is discussed in Section ???. Old and new examples of shared matings are described in Section 5, and particular sequences with arbitrarily high multiplicity are obtained in Sections 6 and 7. Epstein has given various examples of mating discontinuity, which are described in Section 8, and a simple new construction is presented.

Hyperbolic quadratic rational maps f are classified as follows according to Rees [?] and Milnor [?]:

B or II is **bitransitive**: both critical points are in the same cycle of Fatou components but not in the same component.

C or III is a **capture**: one critical point is in a strictly preperiodic Fatou component.

D or IV has **disjoint** cycles of Fatou components.

E or I is **escaping**: both critical orbits converge to a fixed point within the only Fatou component.

Now each hyperbolic component of type B, C, D contains a unique postcritically finite map up to normalization, but there is no such map of type E. While hyperbolic anti-matings may be of type B, C, or D [?], every hyperbolic mating is of type D. The converse is false according to Ben Wittner [?]:

Example 3.1 (Wittner)

Xw34

There is a unique real quadratic rational map of the form $f_w(z) = (z^2 + a)/(z^2 + b)$, such that 0 is four-periodic and ∞ is three-periodic; approximately $a = -1.3812$ and $b = -0.3881$. This map is not a geometric mating of quadratic polynomials.

Proof: Any mating $f \simeq P \amalg Q$ has this branch portrait, if and only if P is four-periodic and Q is three-periodic. Wittner determined all combinations numerically and found them to be different from f_w . Alternatively, show combinatorially that all of these matings have periodic Fatou components with common boundary points; this is obvious when P or Q is of satellite type. Otherwise P or \bar{P} is the Kokopelli at $\gamma_M(3/15)$ and Q is the Airplane — then four-periodic Fatou components are drawn together pairwise by ray-connections through the two-cycle of the Airplane. On the other hand, for f_w no closed Fatou components in the same cycle meet, since the critical orbits are ordered cyclically as $z_3 < w_2 < z_0 < z_2 < w_1 < z_1 < w_0$ on $\mathbb{R} \cup \{\infty\}$. The Julia set \mathcal{J}_w is a Sierpinski carpet in fact [?]. ■

The characterization of matings by an equator is a folk theorem going back to Thurston; it was proved in [?, ?] under similar assumptions. Statement and proof require some standard notions from Thurston theory, see [?, ?, ?, ?].

Theorem 3.2 (Thurston–Lévy–Wittner)

TmeqW

Suppose f is a postcritically finite rational map of degree $d \geq 2$. Then f is combinatorially equivalent to a formal mating $g = P \sqcup Q$, if and only if it has an **equator** γ : a simple closed curve with the property that $\gamma' = f^{-1}(\gamma)$ is connected and homotopic to γ relative to the postcritical set, traversed in the same direction.

Proof: By construction, a formal mating $g = P \sqcup Q$ has the equator S^1 . So if f is combinatorially equivalent to g , with $\psi_0 \circ g = f \circ \psi_1$, then $\gamma = \psi_0(S^1)$ is homotopic to $\gamma' = f^{-1}(\gamma) = \psi_1(S^1)$. Conversely, when f has an equator, it is equivalent to a Thurston map \hat{g} with $\hat{g}(z) = z^d$ for $z \in S^1$. So \hat{g} is a formal mating of two topological polynomials \hat{P} and \hat{Q} . Suppose \hat{P} is obstructed, thus f is obstructed as well, then it would be a flexible Lattès map with four postcritical points. Now \hat{P} and \hat{Q} together have six postcritical points; since \hat{P} has at least four, \hat{Q} has at most two, so \hat{Q} and f have a periodic critical point. But Lattès maps have only preperiodic critical points, so \hat{P} and \hat{Q} are unobstructed in any case. By the Thurston Theorem, there are equivalent polynomials P and Q , which are determined

uniquely by requiring them monic, centered, and with suitable asymptotics of the 0-ray under the equivalence. Now \hat{g} is equivalent to the formal mating $g = P \sqcup Q$. ■

Remark 3.3 (Equator and pseudo-equator)

Rpeq

1. Suppose $f \cong P \amalg Q$ is a postcritically finite geometric mating. If f is hyperbolic, it is combinatorially equivalent to the formal mating $g = P \sqcup Q$, so it has an equator. If f is not hyperbolic, there may be identifications from postcritical ray-equivalence classes, such that g is obstructed and f is combinatorially equivalent to an essential mating \tilde{g} . Then f does not have an equator corresponding to this representation as a mating.

2. When P and Q have only preperiodic critical points, the essential mating \tilde{g} and the geometric mating f may have a **pseudo-equator**, which passes through all postcritical points; see [?, ?] for the definition. The equator of g is deformed to a pseudo-equator of \tilde{g} , if and only if there are at most direct ray connections between postcritical points. Conversely, when f has a pseudo-equator γ , each pseudo-isotopy from γ to $f^{-1}(\gamma)$ determines a pair of polynomials P, Q with $f \simeq P \amalg Q$.

3. A Thurston map g is **expanding**, if there is a curve C through the postcritical points, such that its n -th preimages form a mesh with maximal diameters going to 0. See [?, ?, ?] for other notions of expansion. According to [?], some iterate g^n has a pseudo-equator and it is equivalent to a topological mating. A finite subdivision rule may be used to define an expanding map [?]; for an essential mating with a pseudo-equator, Wilkerson [?] constructs a subdivision rule from the Hubbard trees.

4 Captures and matings

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Captures and precaptures are ways to construct a Thurston map by shifting a critical value to a preperiodic point; we shall see that precaptures are related to matings with preperiodic polynomials in fact.

4.1 Hyperbolic and repelling-preperiodic captures

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These constructions rely on the concept of shifting or pushing a point from a to b along an arc C . This means that a homeomorphism φ is chosen, which is the identity outside off a tubular neighborhood of C , and such that $\varphi(a) = b$. We may assume that an unspecified point close to a is mapped to a and b is mapped to an arbitrary point nearby.

Proposition 4.1 (and definition)

Phrc

Suppose P is a postcritically finite quadratic polynomial and $z_1 \in \mathcal{K}_P$ is preperiodic and not postcritical. Let the new postcritical set be $P_g = P_P \cup \{P^n(z_1) \mid n \geq 0\}$. Consider an arc C from ∞ to z_1 not meeting another point in P_g and choose a homeomorphism φ shifting ∞ to z_1 along C , which is the identity outside off a sufficiently small neighborhood of C . Then:

- $g = \varphi \circ P$ is well-defined as a quadratic Thurston map with postcritical set P_g . It is a **capture** if z_1 is eventually attracting and a **precapture** in the repelling case.
- The combinatorial equivalence class of g depends only on the homotopy class of the arc C .

Proof: By construction, g is a postcritically finite branched cover, when the neighborhood of C does not include any postcritical point except z_1 . Note that the preimages of z_1 under P are mapped to some arbitrary point by g , so if z_1 was periodic or postcritical, g would not be well-defined. Finally, if we have two different homeomorphisms φ and φ' along the same curve or along two homotopic curves, then $g' = (\varphi' \circ \varphi^{-1}) \circ g$ and the homeomorphism $\varphi' \circ \varphi^{-1}$ is isotopic to the identity, since the appended path $C' \cdot C^{-1}$ is contractible relative to $P_g \setminus \{z_1\}$. ■

Consider the following applications and possible generalizations:

- If a capture $g = \varphi \circ P$ is combinatorially equivalent to a rational map f , this gives a hyperbolic map of capture type. Let us say that f is a **Wittner capture**, if the capture path C is homotopic to a rational external ray followed by an internal ray of P ; this construction is due to Ben Wittner [?] and Mary Rees [?]. Note that Rees denotes only Wittner captures as captures, while general captures are called maps of type III. Maps of this type are never matings, but they may have a representation as an anti-mating [?].
- Precaptures along external rays are related to matings in the following Section 4.2.
- Precaptures apply not only to polynomials P , but to rational maps in general as long as the other critical orbits are finite. This construction provides a finite **regluing** followed by a possible combinatorial equivalence. In a more general situation, a countable regluing is followed by a semi-conjugation [?, ?].
- **Recapture** means that the finite critical value $P(0)$ is shifted to a preimage of 0, resulting in a Thurston map equivalent to a hyperbolic polynomial. Relations to internal addresses and to Dehn twisted maps are discussed in [?].

Initialization 4.2 (Captures and precaptures)

Thrc

Consider a capture or precapture $g = \varphi \circ P$ according to Proposition 4.1. Then the Thurston Algorithm is implemented by pulling back a path in moduli space, which is initialized as follows: normalize P such that the critical points are $0, \infty$ and another point in $P_g \setminus \{z_1\}$ is 1. For $0 \leq t \leq 1$, $x_1(t)$ moves from ∞ to z_1 along C , while all of the other marked points stay fixed.

Under a non-conjugate-limbs condition, Wittner captures are unobstructed [?] and precaptures along external rays have only obstructions satisfying the assumptions of Theorem ??; see below. So the sequence of rational maps converges to a rational map f , unless the orbifold of f is of type $(2, 2, 2, 2)$: then the sequence does not converge in general, but it might converge for a special choice of C .

Proof: Note that when the preperiod of z_1 is one, the corresponding periodic point satisfies $\psi_t(-z_1) = -x_1(t)$ only for $t \geq 1$. Since $\varphi^{-1} \circ g = P \circ \text{Id}$ and P is holomorphic, we have $[\text{Id}] = \sigma_g([\varphi^{-1}])$ and we may initialize the Thurston Algorithm with a path ψ_t from $\psi_0 = \varphi^{-1}$ to $\psi_1 = \text{Id}$. Now $\varphi^{\pm 1}$ is the identity outside off a small neighborhood of C , so ψ_t can be chosen such that it moves $x_1(t) = \psi_t(z_1)$ from $\varphi^{-1}(z_1) = \infty$ to z_1 along C , and leaves the other marked points untouched. By Proposition ?? the projection from \mathcal{T} to \mathcal{M} defines a suitable initialization to compute the Thurston pullback $\pi(\sigma_g^n)$ from an explicit pullback in moduli space. ■

To illustrate the process of slow capture or precapture, we may also define a sequence or path of images $\psi_t(\mathcal{K}_p)$ of the filled Julia set, which is constant \mathcal{K}_p for $0 \leq t \leq 1$. It will show more and more identifications happening by a piecewise pseudo-isotopy. See also the videos on www.mndynamics.com. A similar initialization is used for Dehn twisted maps; see [?] and the Examples 3.1 and 3.7 in [?].

4.2 Precaptures and matings

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The representation of matings by precaptures along external rays is motivated by remarks in [?, ?]. In the former paper, the boundary of a capture component in V_n is described by matings, which are related to the postcritically finite map of capture type by regluing. This means that the critical value is shifted from ∞ along an external ray followed by an internal ray, and then moved back along an internal ray. So can the mating be constructed by shifting the critical value directly from ∞ to $z_1 = \gamma_p(\theta)$ along the external ray $\mathcal{R}_p(\theta)$? This is true in general when z_1 is preperiodic, not only when it is on the boundary of a hyperbolic component, but we shall not discuss postcritically infinite maps here.

Theorem 4.3 (Matings as precaptures)

TRprec

Suppose P is postcritically finite and θ is preperiodic, such that $q = \gamma_M(-\theta)$ is not in the conjugate limb and $z_1 = \gamma_p(\theta) \in \partial\mathcal{K}_p$ is not postcritical. Then the precapture $g_\theta = \varphi_\theta \circ P$ along $\mathcal{R}_p(\theta)$ is combinatorially equivalent or essentially equivalent to the geometric mating f defined by $P \amalg Q$.

So if $P \amalg Q$ is not of type $(2, 2, 2, 2)$, any implementation of the Thurston pull-back for g_θ gives a converging sequence of rational maps; e.g., Initialization 4.2 applies. The normalization $\beta_p = 1$ ensures $f(1) = 1$. Note that the precapture does not work if both P and Q are hyperbolic; then there is an alternative construction with two paths [?]. When only one of the two polynomials is hyperbolic, then either $P \amalg Q$ or $Q \amalg P$ is a precapture. And when both are critically preperiodic, then both $P \amalg Q$ and $Q \amalg P$ are precaptures, unless a critical point is iterated to the other critical point: then ∞ shall be iterated to 0. — By choosing precaptures along homotopic external rays, examples of shared matings are obtained in [?].

Recall the notation g and \tilde{g} for the formal mating and the essential mating; we shall see below that there is an essential precapture \tilde{g}_θ as well. Before showing $\tilde{g}_\theta \sim \tilde{g}$ let us consider a few examples, to see how identifications happen and why they may happen in different ways for g and g_θ :

- When $g = 9/56 \sqcup 1/4$, so $\theta = 3/4$, there are no postcritical identifications: $\tilde{g} = g$ and $\tilde{g}_\theta = g_\theta$. The precapture can be constructed from the formal mating by shifting all postcritical points in $\varphi_\infty(\mathcal{K}_q)$ to $\varphi_0(\mathcal{K}_p)$ along external rays, so g_θ and g are combinatorially equivalent.
- In reverse order we have $\tilde{g} = g = 1/4 \sqcup 9/56$ again, but $\tilde{g}_\theta \neq g_\theta$ for $\theta = 47/56$ and $p = \gamma_M(1/4)$. Now $g_\theta(\infty)$ has preperiod and period three, but $\tilde{g}_\theta(\infty)$ has period one. The shift φ_θ creates a subset of the lamination with angle θ in the exterior of \mathcal{K}_p , so there is a triangle connecting $3/7, 5/7, 6/7$ with a homotopic preimage under g_θ ; pinching the surrounding Lévy-cycle gives \tilde{g}_θ .

- The converse happens for $g = 1/4 \sqcup 3/14$, so $p = \gamma_M(1/4)$ and $\theta = 11/14$. Now both $q = \gamma_M(3/14)$ and $g(\infty)$ have preperiod one and period three, while $\tilde{g} \neq g$ has period one. But this identification is immediate in the precapture $g_\theta = \tilde{g}_\theta$, since $z_1 = -\alpha_p$.
- Both phenomena happen at the same time for $g = 3/14 \sqcup 3/14$, so $\theta = 11/14$. In g_θ the 3-cycle of P is collapsed by a triangle in the exterior, while the 3-cycle of Q is identified with α_p immediately. We have $\tilde{g}_\theta \neq g_\theta \not\sim g \neq \tilde{g}$.

For longer ray connections, there may be a similar splitting of branch points and similar immediate identifications, but otherwise the precapture can be understood in terms of the same ray-equivalence classes as the formal mating:

Proof of Theorem 4.3: Denote by X the union of all postcritical ray-equivalence classes of the formal mating $g = P \sqcup Q$. Define another Thurston map g^θ by shifting the critical value $\varphi_\infty(q)$ to $\varphi_0(z_1)$ along \mathcal{R}_θ , without modifying g on X . Consider the extended Hubbard tree $T_p \subset \mathcal{K}_p$, which consists of regular arcs connecting the postcritical points of g_θ . Then $g_\theta : T'_p \rightarrow T_p$, where $T'_p = T_p$ except for a slight detour at $P^{-1}(0)$. We may assume that $g^\theta \circ \varphi_0 = \varphi_0 \circ g_\theta$ in a neighborhood of T_p . So the two maps are combinatorially equivalent, even if we mark the critical point ∞ in addition, since all other marked points are contained in T_p and T_p is connected.

Now consider a path of Thurston maps g_t , such that postcritical points of P stay fixed in $\varphi_0(\partial\mathcal{K}_p)$ and all postcritical points of Q move from $\varphi_\infty(\partial\mathcal{K}_q)$ to $\varphi_0(\partial\mathcal{K}_p)$ along external rays of g . This deformation is a kind of two-sided pseudo-isotopy from g to g^θ , since marked points may collapse in different ways on both ends, while each component of X is invariant under each g_t . By collapsing all components of X to points and modification at preimages, equivalent quotient maps are obtained for all g_t , in particular for g and g^θ , where postcritical points have been identified already in different ways. So we know that $\tilde{g}^\theta = \tilde{g} \sim f$ and we may consider \tilde{g}^θ as an essential map in the sense of Definition ??, with Γ consisting of loops around those trees in X , which contain at least two postcritical points of g^θ . So g^θ is essentially equivalent to f , combinatorially equivalent if $\Gamma = \emptyset$, and the same applies to the original precapture g_θ . ■

5 Shared matings

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A shared mating is a geometric mating with different representations, $P_1 \amalg Q_1 \simeq f \simeq P_2 \amalg Q_2$ with $P_1 \neq P_2$ or $Q_1 \neq Q_2$. There are the following examples of shared matings, and techniques for constructing them:

- Wittner [?] introduced the notion of shared matings and discussed them for V_3 in particular. A simple example is given by the geometric mating of Airplane and Rabbit, which is affine conjugate to the geometric mating of Rabbit and Airplane, $A \amalg R \simeq R \amalg A$. (Moreover, it is conjugate to a symmetric map, which is not a self-mating.) Since the two polynomials are interchanged, this example is called the Wittner flip. It can be explained by finding two different equators, which has a few generalizations:

- Exall [?] constructs pairs of polynomials P, Q with $P \amalg R \simeq Q \amalg A$ from a second equator. Using symbolic dynamics, this can be done algorithmically.
- Rees [?] uses symbolic dynamics again to obtain unboundedly shared Airplane matings. The period grows exponentially with the multiplicity.
- Denote the rabbit of rotation number k/n by R . There are $n - 2$ primitive hyperbolic polynomials Q of period n , such that \overline{Q} has a characteristic angle from the cycle of α_r . Then the rational map $f \cong R \amalg Q$ has a cluster cycle: both n -cycles of Fatou components have a common boundary point, which is a fixed point corresponding to α_r . Tom Sharland [?, ?] has shown that f is determined uniquely by the rotation number and the relative displacement of the critical orbits; f has precisely two representations as a mating, which are of the form $f \cong R \amalg Q \simeq P \amalg R$.

When f is a Lattès map, different representations are known except in the case c) of $1/6 \amalg 1/6$. The Shishikura Algorithm can be used to identify the particular map f in the case of type $(2, 2, 2, 2)$, and we have only one quadratic map of type $(2, 4, 4)$. Combinatorial arguments show that there are basically nine, respectively three, matings of these types; see Sections 4 and 6 in [?].

- Case a) of type $(2, 2, 2, 2)$ is $1/4 \amalg 1/4 \simeq 23/28 \amalg 13/28 \simeq 13/28 \amalg 23/28 \simeq 53/60 \amalg 29/60 \simeq 29/60 \amalg 53/60$.
- Case b) of type $(2, 2, 2, 2)$ is given by $1/12 \amalg 5/12 \simeq -1/12 \amalg 5/12$.
- Case d) of type $(2, 2, 2, 2)$ is $1/6 \amalg 5/14 \simeq 5/14 \amalg 1/6 \simeq 3/14 \amalg 3/14 \simeq 3/14 \amalg 1/2 \simeq 1/2 \amalg 3/14 \simeq 5/6 \amalg 1/2 \simeq 1/2 \amalg 5/6$.
- Type $(2, 4, 4)$ is given by $\pm 1/4 \amalg 1/2 \simeq 5/12 \amalg \pm 1/6 \simeq 13/28 \amalg \pm 3/14$.

The following technique for producing shared matings is based on the representation of matings as repelling-preperiodic captures according to Theorem 4.3.

Proposition 5.1 (Shared matings from precaptures)

Pshprec

Suppose $P(z) = z^2 + p$ is geometrically finite, with $p \neq -2$, $p \neq 1/4$, and p not in the main cardioid. There are countably many pairs of preperiodic angles θ_1, θ_2 such that: the corresponding dynamic rays land together at a preperiodic pinching point $z_1 \in \partial\mathcal{K}_p$, which is not postcritical and not in the same branch at α_p as p , and the branch or branches of z_1 between these rays do not contain postcritical points of P or iterates of z_1 . Then we have $P \amalg Q_1 \simeq P \amalg Q_2$ with $q_i = \gamma_M(-\theta_i)$. Moreover, $P \amalg Q_1 \cong P \amalg Q_2$ if β_p is not between these rays.

Proof: We need to exclude $p = 1/4$ and the main cardioid, because \mathcal{K}_p would have no pinching points, and $p = -2$, because rays landing together at the interval \mathcal{K}_{-2} are never homotopic with respect to the postcritical set. If P is postcritically finite, Proposition 4.1 shows that the precaptures $\varphi_{\theta_1} \circ P$ and $\varphi_{\theta_2} \circ P$ are combinatorially equivalent. So the canonical obstructions and the essential maps are equivalent as well. According to the proof of Theorem 4.3, given in [?], the essential maps are equivalent to the geometric matings. By continuity according to Section ??, the result extends to geometrically finite P : ■

- The example $11/24 \amalg 13/56 \cong 11/24 \amalg 15/56$ enjoys the following property: the latter mating has an equator and a simple pseudo-equator, while the former does not have either.
- As another example, consider $p = \gamma_M(59/240)$ and $q = \gamma_M(63/240)$. Applying this construction to P and to Q gives $P \amalg P \cong P \amalg Q$ and $Q \amalg P \cong Q \amalg Q$. Here the first and second polynomials may be interchanged on both sides, so we have four representations of the same rational map; in particular there are shared self-matings $P \amalg P \cong Q \amalg Q$, and the flipped matings $P \amalg Q \cong Q \amalg P$.
- When P is the Basilica, all pinching points are preimages of α_p . Since none of these is iterated behind itself, shared matings are obtained from any pinching point z_1 , which is not α_p or behind it. Dudko [?] has shown that these are the only shared Basilica matings, since the parameter space is described as a mating of \mathcal{M} and \mathcal{K}_p . The simplest example is given by $P \amalg (z^2 \pm i)$: the geometric matings are distinct and complex conjugate, and both affine conjugate to $\frac{z^2+2}{z^2-1}$. The example $P \amalg 5/24 \simeq P \amalg 7/24$ is illustrated with a video of slow mating on www.mndynamics.com. Aspenberg [?] constructs the semi-conjugation from the Basilica to the rational map, beginning with the Boettcher map; in this alternative approach, shared matings are obtained from a non-unique labeling of Fatou components by bubble rays.
- Shared matings in the family of Chebychev maps are discussed in Section 6. In certain cases, lower bounds on the multiplicity are obtained from homotopic rays according to Proposition 5.1, or upper bounds are obtained directly.
- When z_1 is a branch point of \mathcal{K}_p , there may be more than two parameters q_i . In Theorem 7.1 of Section 7, unboundedly shared Airplane matings with small preperiods and periods are constructed. Although the Airplane does not contain any branch point, this is achieved by choosing q_i with a common branch point in \mathcal{K}_q .
- If f is a critically preperiodic rational map of degree $d \geq 2$ with three or four postcritical points, a pseudo-equator may produce several unmatings by choosing different pseudo-isotopies to its preimage [?]. A higher multiplicity is obtained when there are degenerate critical points, or when a critical point is mapped to another one. Probably the only quadratic example is the Lattès map of type (2, 4, 4). See [?] for related results on NET maps.

Remark 5.2 (Finite multiplicity)

Rfmsm

If f is a postcritically finite quadratic rational map, can there be infinitely many representation as a mating $f \simeq P \amalg Q$?

- When f is hyperbolic, there are only finitely many candidates for P and Q , since there are only finitely many quadratic polynomials with a given superattracting period.
- When one critical point is periodic and one is preperiodic, finiteness is not obvious. For a specific family, finiteness is shown in Theorem 6.2 of the following section, using similar techniques as in the Lattès case.
- When both critical points are preperiodic, finiteness is shown for Lattès maps

in [?]. Probably the techniques can be applied to a few other examples of small preperiod and period, but a general proof shall be harder.

6 Shared matings in the Chebychev family

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Let us define a Chebychev map as a quadratic rational map of the form $f(z) = f_a(z) = \frac{z^2 - a - 2}{z^2 + a}$, $a \neq -1$, for which $f(\infty)$ is pre-fixed: $\infty \Rightarrow 1 \rightarrow -1 \uparrow$. This family contains matings with the Chebychev polynomial in particular:

Proposition 6.1 (Chebychev maps as matings)

Pchmch

Suppose $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ are geometrically finite and not in conjugate limbs of the Mandelbrot set \mathcal{M} . Then the geometric mating is affine conjugate to a Chebychev map, $f_a \simeq P \sqcup Q$, if and only if P and Q are one of the following forms:

- Q is the Chebychev polynomial $Q(z) = z^2 - 2$ and p is not in the $1/2$ -limb of \mathcal{M} .
- p is in the k/r -limb of \mathcal{M} , and $q = \gamma_{\mathcal{M}}(-\theta)$, where θ is one of the r angles at $-\alpha_p \in \mathcal{K}_p$ (which depend only on k/r).
- For a rotation number $k/r \neq 1/2$, denote the angles of the k/r -wake by θ_{\pm} and let $\theta = (\theta_- + \theta_+)/2$ be the unique angle of preperiod 1 and period r in that limb. If $q = \gamma_{\mathcal{M}}(\theta)$, then P must be in the closed wake of the primitive hyperbolic component Ω with the root $\gamma_{\mathcal{M}}(-2\theta)$.

The Petersen transformation [?] maps symmetric rational maps to Chebychev maps, such that self-matings are mapped to Chebychev matings; see also Remark 4.4 in [?]. In the previous section the example of shared self-matings $59/240 \sqcup 59/240 \cong 63/240 \sqcup 63/240$ was obtained from Proposition 5.1; now the Petersen transformation gives the shared Chebychev mating $59/240 \sqcup 1/2 \cong 63/240 \sqcup 1/2$.

Proof of Proposition 6.1: As explained in Figure 1, instead of saying that angles of $z \in \mathcal{K}_p$ and $w \in \mathcal{K}_q$ are complex conjugate, we may say that $z \in \mathcal{K}_p$ shares an angle with $\bar{w} \in \mathcal{K}_{\bar{q}}$, or connect $\mathcal{K}_{\bar{p}}$ to \mathcal{K}_q as well. In the formal mating $g = P \sqcup Q$, the ray-equivalence class of $g^2(\infty)$, corresponding to $\bar{Q}^2(0) = \bar{Q}(\bar{q})$, is fixed. By Proposition ??, it must contain a fixed point of P or \bar{Q} . If this is β_p or $\beta_{\bar{q}}$, the fixed class is the 0-ray and $\bar{Q}(\bar{q}) = \beta_{\bar{q}}$, which is case a).

b) Now suppose that $\bar{Q}(\bar{q})$ is in the same ray-equivalence class as α_p and $p \in \mathcal{M}_{k/r}$. Then the critical value \bar{q} is connected to $-\alpha_p$. This connection must be direct, since \mathcal{K}_p does not contain another pinching cycle of ray period r . So \bar{q} shares an external angle with $-\alpha_p$ and all of these angles may occur, since none is in the same sector at α_p as the critical value p , and q is not in the conjugate limb. The r angles belong to different Misiurewicz points in fact, since otherwise some $P \sqcup Q$ would have a closed ray connection.

c) Consider $q \in \mathcal{M}_{k/r}$ and \bar{P} such that $Q(q)$ is in the same ray-equivalence class as α_q . The points are not equal, because the preperiod would have to be $\geq r > 1$. So the ray connection must have length 2, since length ≥ 4 would require additional pinching cycles of ray period r in $\mathcal{M}_{k/r}$. Thus q has the external angle θ defined above, and $\mathcal{K}_{\bar{p}}$ must contain a pinching cycle of period and ray period r , which connects the cycle of $2\theta = \theta_- + \theta_+$ to that of θ_{\pm} . This cycle of $\mathcal{K}_{\bar{p}}$ persists from a primitive hyperbolic component $\bar{\Omega}$ before \bar{p} .

It remains to show that $\bar{\Omega}$ exists and is unique. In the dynamic plane of Q , the r rays landing at α_q define r sectors W_1, \dots, W_r with $0 \in W_r$ and $q \in W_1$, such that Q is a conformal map $W_1 \rightarrow W_2 \rightarrow \dots \rightarrow W_{r-1} \rightarrow W_r$ and the sectors are permuted with rotation number k/r . The external rays with angles $2^{i-1}\theta_{\pm}$ bound W_i for $1 \leq i \leq r$. Now W_i contains $2^{i-1}\theta$ as well for $2 \leq i \leq r-1$ and W_r has both $2^{r-1}\theta$ and $2^r\theta = -\theta$. For $r \geq 3$ it follows that 2θ has exact period r . We are looking for a primitive orbit portrait [?] connecting each angle in $\{2^i\theta \mid 1 \leq i \leq r\}$ to a unique angle in $\{2^i\theta_- \mid 1 \leq i \leq r\} = \{2^i\theta_+ \mid 1 \leq i \leq r\}$.

Starting in W_r , connect $2^{r-1}\theta$ to either $2^{r-1}\theta_-$ or to $2^{r-1}\theta_+$, such that $2^r\theta$ is not separated from the other angles. Pull the connection back until 2θ is connected to $2\theta_-$ or $2\theta_+$. The complement of the $r-1$ disjoint small sectors is connected, so we can connect the remaining angles $2^r\theta$ and θ_+ or θ_- as well. This construction gives a valid orbit portrait and defines $\bar{\Omega}$, which has the external angles 2θ and $2\theta_-$ or $2\theta_+$. Note that it is a narrow component, i.e., its angular width is $1/(2^r - 1)$ and there is no component of period $\leq r$ behind $\bar{\Omega}$. To show that $\bar{\Omega}$ is unique, suppose we had started by connecting $2^{r-1}\theta$ with an angle not bounding W_r and pulled it back. This pullback would follow the rotation number k/r as well and the small sectors would overlap, the leaves would be linked. ■

Case b) provides maps from limbs of \mathcal{M} to the Chebychev family, which are partially shared according to Proposition 5.1: e.g., for P geometrically finite in the $1/2$ -limb, consider the geometric matings corresponding to $P \amalg \pm 1/6$, i.e. $p \mapsto f_a \simeq P \amalg \pm 1/6$. These two maps agree on the small Mandelbrot set of period 2, but in general do not agree on its decorations. Likewise, for p in the $1/3$ -limb, we have three maps corresponding to $P \amalg 3/14$, $P \amalg 5/14$, and $P \amalg 13/14$, which agree on the small Mandelbrot set of period 3. In the decorations, two of the maps may agree on certain veins, but in general the third one will be different: the relevant rays are no longer homotopic. Note that according to case c), some of these Chebychev maps are represented by $\tilde{P} \amalg 3/14$ as well, with \tilde{p} in the Airplane wake. In particular, we have $1/7 \amalg 3/14 \cong 1/7 \amalg 5/14 \simeq 1/7 \amalg 13/14 \simeq 3/7 \amalg 3/14$. Under the Petersen transformation mentioned above, this Chebychev map is the image of $1/7 \amalg 3/7 \simeq 3/7 \amalg 1/7$, which is a symmetric map but not a self-mating.

Theorem 6.2 (Chebychev maps as shared matings)

Tchmun

Matings $P \amalg Q$ in the Chebychev family with hyperbolic P have non-uniformly bounded multiplicity:

1. Suppose $f = f_a$ is a Chebychev map, such that $z = 0$ is n -periodic. Then there are at most a finite number of representations $f_a \simeq P \amalg Q$.
2. For each rotation number k/r , there is a unique Chebychev map $f = f_a$, such that $z = 0$ is r -periodic and the fixed point $-1 = f^2(\infty)$ is a common boundary point of the r immediate basins, which are permuted with rotation number k/r . This map has precisely $r+1$ realizations as a geometric mating, $f \simeq P \amalg Q$, when $r \geq 3$; for $r = 2$ there are only 2 representations.

Proof: 1. P will be n -periodic, so there are only finitely many possibilities for P . We must see that r is bounded in cases b) and c). But in both cases we have $r \leq n$, since the wakes of period r are narrow: in case b) this is a basic property of limbs, and in case c) it was noted in the proof of Proposition 6.1.

2. In case a) of Proposition 6.1, $z = -1$ corresponds to the ray-equivalence class of angle 0, which does not touch a hyperbolic component of P . In cases b)

and c), the rotation number at -1 is precisely k/r , so the value of k/r from the proposition must be the same as in the hypothesis of the theorem; case c) is excluded for $k/r = 1/2$. In both cases, there is only one hyperbolic component of period r in the limb or wake. It remains to show that f_a is unique, so that the $r + 1$ (or two) matings actually give the same map. Intuitively, this follows from the fact that the hyperbolic component of f_a bifurcates from the hyperbolic component where -1 is attracting; the multiplier map with $\rho = \frac{-4}{a+1}$ is injective for $|a + 1| \geq 4$. It can be proved by Thurston rigidity, since there is a forward-invariant graph connecting the postcritical points, which depends only on k/r up to isomorphy. So all possible maps f_a are combinatorially equivalent, affine conjugate, and equal. — Note that the case of $k/r = 2/5$ was discussed in the Introduction and in Figure 1. ■

7 Unboundedly shared Airplane matings

3u

Denoting the Rabbit by R and the Airplane by A , we have seen in the previous Section 6 that $R \amalg 3/14 \cong R \amalg 5/14 \simeq R \amalg 13/14 \simeq A \amalg 3/14$. This example belongs both to the Chebychev family and to the family V_3 with a 3-periodic critical point. Unboundedly shared matings were obtained in Theorem 6.2.2 by increasing both the period of the hyperbolic polynomial P and the ray period of the Misiurewicz polynomial Q . Another example is obtained below, where Q is always the Airplane, and the preperiod of P is unbounded. The proof will be a simple application of Proposition 5.1 again. Airplane matings with unbounded multiplicity are due to Rees [?] with hyperbolic polynomials P , such that the period of P grows exponentially with the multiplicity.

Theorem 7.1 (Unboundedly shared Airplane matings)

Tunbampp

For the Airplane q and $n = 3, 5, 7, \dots$, there are n Misiurewicz parameters p_*, p_2, \dots, p_n such that the geometric matings agree, $f \cong P_i \amalg Q$ for all $i = *, 2, \dots, n$. Here all p_i have preperiod $n + 1$, p_* has period 1 and p_2, \dots, p_n have period n ; so $f(\infty)$ has preperiod $n + 1$ and period 1. The statement remains true for large n , when q is any geometrically finite parameter behind $\gamma_M(5/12)$ and before the Airplane. E.g., q may be the Misiurewicz point $\gamma_M(41/96)$ as well.

Proof: Denote the Airplane parameter by q and fix $n \in \{3, 5, 7, \dots\}$; let c be the first center of period n behind the Misiurewicz point $\gamma_M(5/12)$. The orbit of the characteristic point z_1 is ordered as

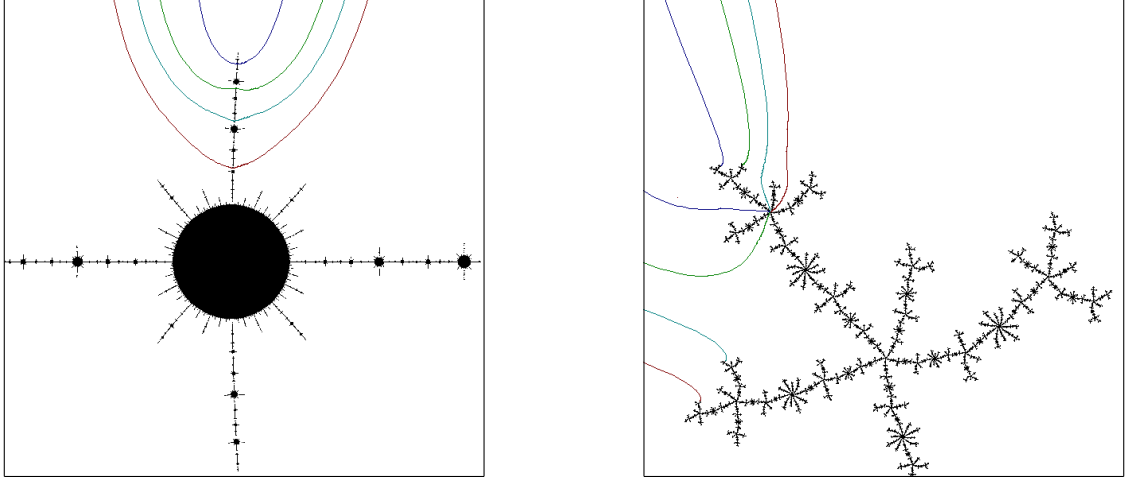
$$\begin{aligned} z_1 &< \gamma_c(5/12) < z_{n-1} < z_{n-3} < \dots < z_6 < z_4 < \alpha_c < \\ &< z_3 < z_5 < \dots < z_{n-2} < z_n < 0 < -\alpha_c < z_2 ; \end{aligned} \tag{1}$$

shko

the critical orbit (z_i^*) is similar with $z_n^* = 0$. This ordering is well-known from discussions of Šarkovskii combinatorics. It can be checked with dynamic angles as follows: first, note that the order of the critical orbit is compatible with the assumption that $f_c : [z_1^*, 0] \rightarrow [z_1^*, z_2^*]$ is strictly decreasing and $f_c : [0, z_2^*] \rightarrow [z_1^*, z_3^*]$ is strictly increasing, so this defines a unique real polynomial. Let Θ_1 be the larger angle at z_1 and denote its iterates under doubling by Θ_i . Then

$$\begin{aligned} 0 &< \Theta_2 < 1/6 < \Theta_n < \Theta_{n-2} < \dots < \Theta_5 < \Theta_3 < 1/3 < \\ &< 1/2 < \Theta_1 < 7/12 < \Theta_{n-1} < \Theta_{n-3} < \dots < \Theta_6 < \Theta_4 < 2/3 < 1 , \end{aligned} \tag{2}$$

shka



Fmult

Figure 2: Consider the formal mating $g = P \sqcup Q$, with the Airplane \mathcal{K}_q shown rotated on the left, and \mathcal{K}_p on the right for some p in the $2/5$ -limb. According to the proof of Theorem 7.1, there are eight angles $\theta_2, \dots, \theta_5, \theta'_2, \dots, \theta'_5$, such that $-\theta_i$ and $-\theta'_i$ land together at the Airplane $\partial\mathcal{K}_q$, while θ_i land together at $\partial\mathcal{K}_p$. So the eight rays belong to a preperiodic ray-equivalence class of diameter four; actually there are two more rays crossing the Airplane on the real axis. Now there are five parameters $p = p_*, p_2, \dots, p_5$, such that this ray-equivalence class contains the critical value p , and it is shown that the corresponding matings define the same rational map f .

since the derivative of the real polynomial $f_c(z) = z^2 + c$ is negative for $z < 0$ and then f_c swaps the lower and upper half-planes. Reading off binary digits gives $\Theta_1 = .100101\dots 010$, which is the largest n -periodic angle less than $7/12 = .10\overline{01}$. Reversing these arguments, it follows that the center defined by $\gamma_M(\Theta_1)$ is real and the orbit is as given by (1). Each z_i has two external angles, Θ_i and $1 - \Theta_i$. Note that $f_c : [0, \beta_c] \rightarrow [c, \beta_c]$ is increasing; taking preimages of z_i with respect to this branch gives strictly preperiodic points except for z_3 , which has a periodic preimage z_2 on the positive half-axis.

Now consider any parameter p in the limb with rotation number k/n , $k = (n - 1)/2$. The wake is bounded by $0 < \theta_- < \theta_+ < 1/3$. We have $\theta_+ = .\overline{0101\dots 010}$, since this is the largest n -periodic angle less than $1/3 = .\overline{01}$, or by sketching an n -Rabbit. So $\theta_+ = \Theta_3$ and $\theta_- = \Theta_5$; note that $\Theta_1 = 1/2 + \theta_+/4$ is an instance of the Douady Magic Formula from Proposition ?? . — The critical value p of $f_p(z) = z^2 + p$ is in the sector at α_p bounded by the dynamic rays with the angles θ_{\pm} . This sector is mapped injectively for $n - 1$ iterations; the image sector contains $0, -\alpha_p$, and a unique point in $f_p^{-1}(-\alpha_p)$. Thus the original sector contains unique preimages of α_p with preperiods n and $n + 1$, respectively. Denote the angles of the latter by $\theta_1 < \dots < \theta_n$. Under n iterations, these are mapped to angles at $-\alpha_p$, such that θ_1 gives the smallest angle in $[1/2, 1]$ and θ_n gives the largest angle in $[0, 1/2]$. So under $n + 1$ iterations, θ_1 is mapped to $\theta_+ = \Theta_3$ and θ_n is mapped to $\theta_- = \Theta_5$.

Next, let us look at the Airplane Julia set \mathcal{K}_q with $Q(z) = f_q(z) = z^2 + q$. As the parameter was shifted from c to q , the n -periodic points with angles Θ_i moved holomorphically; in particular the pre-characteristic points corresponding to $\pm z_n$ bound an interval containing the real slice of the Airplane Fatou component around 0. Consider the Fatou component of f_c at z_3 ; it defines an interval in \mathcal{K}_q , which

contains a unique preperiodic component Ω of preperiod $n - 3$. Its largest antenna in the upper halfplane has angles in a subset of $[\Theta_5, \Theta_3] = [\theta_-, \theta_+]$. Since f_q^{n-3} maps it to the largest antenna on the upper side of the Fatou component around 0, f_q^{n-2} maps it behind the component around q . Then it is behind the component around $f_q(q)$, then to the right of the component at 0, and finally we see that f_q^{n+1} maps the antenna of Ω to the interval $(\gamma_q(4/7), \beta_q]$. Denote by x_i the preimage of the n -periodic point with angle Θ_i , then x_3 has preperiod n and the others have preperiod $n + 1$. On the other hand, the angles θ_i are the only angles of preperiod $n + 1$ in (θ_-, θ_+) that are iterated to some Θ_j . Recalling that θ_1 is iterated to $\theta_+ = \Theta_3$, we see that each θ_i with $i \neq 1$ lands at some x_j with $j \neq 3$. Denote the other angle by $\theta'_2, \dots, \theta'_n$; it is in (θ_-, θ_+) as well, since the antenna is contained in an open half-strip bounded by these rays and a real interval.

Finally, define the Misiurewicz parameters $p_* = \gamma_M(\theta_1) = \dots = \gamma_M(\theta_n)$ and $p_i = \gamma_M(\theta'_i)$, $i = 2, \dots, n$. Now p_* is of α -type by construction, so it has preperiod $n + 1$ and period 1. The p_i are endpoints, since there is no other hyperbolic component of period n in the k/n -limb; they are pairwise different in particular. Note that for $i = 2, \dots, n$, the rays $\mathcal{R}_q(-\theta'_i)$ and $\mathcal{R}_q(-\theta_i)$ land together as well and the landing point never returns to this wake, so the two rays are homotopic with respect to its orbit and to the real orbit of q , and the precaptures are equivalent: by Proposition 5.1, the matings $Q \amalg P_i \cong Q \amalg P_*$ agree, as do $P_i \amalg Q \cong P_* \amalg Q$. — For the example of $k/n = 2/5$, Figure 2 shows the rays with angles $-\theta_i, -\theta'_i$ landing pairwise at $\partial\mathcal{K}_q$, and the rays with angles θ_i, θ'_i landing at $\partial\mathcal{K}_{p_*}$, at a preimage of α_{p_*} and at endpoints, respectively.

The landing pattern at $\partial\mathcal{K}_q$ is stable for parameters q between c of period n above and the Airplane, but the relevant antenna will bifurcate when q is too far behind the Airplane. ■

Note that we have constructed n different matings giving the same rational map, but in contrast to Theorem 6.2, no upper bound on the multiplicity is known in this case. — Assuming that the map $\mathcal{M}_{k/r} \rightarrow V_3$, $P \mapsto f \cong P \amalg Q$ is continuous, there will be self-intersections of the image corresponding to these shared matings.

8 Counterexamples to continuity of mating

3c

Geometric mating is not jointly continuous on the subset of $\mathcal{M} \times \mathcal{M}$ where it can be defined. The first three examples below are due to Epstein [?, ?]. Note that all of these techniques involve neutral parameters, and that they do not exclude separate continuity. For specific one-dimensional slices with Q fixed, partial results on continuity have been obtained by Dudko [?] and by Ma Liangang [?].

— *Special thanks to Adam Epstein for explaining unpublished results.*

- Let f_λ be a quadratic polynomial with a fixed point of attracting multiplier λ . For $|\lambda| < 1$, $|\mu| < 1$ there are explicit rational maps $F_{\lambda, \mu} \simeq f_\lambda \amalg f_\mu$. Suppose $\lambda, \mu \rightarrow 1$ tangentially, such that the third multiplier ν is constant. Then if $F_{\lambda, \mu}$ converges to a quadratic rational map, it will depend on ν , so there are oscillating sequences as well. Note that convergence may depend on a normalization allowing the collision of the respective fixed points; in a different normalization, $F_{\lambda, \mu}$ might converge to a map of degree one or to a constant as well.

- Results on shared matings with cluster cycles by Sharland [?, ?] are reported in Section 5. For rotation number $1/n$, we have $f_n \cong R_n \amalg Q_n \simeq P_n \amalg R_n$, where the center parameters correspond to the following roots: $r_n \sim \gamma_M(1/(2^n - 1)) = \gamma_M(2/(2^n - 1))$, $q_n \sim \gamma_M(-3/(2^n - 1)) = \gamma_M(-4/(2^n - 1))$, and $p_n \sim \gamma_M((2^{n-1} - 1)/(2^n - 1)) = \gamma_M(2^{n-1}/(2^n - 1))$. Then $r_n \rightarrow r_0 = 1/4 = \gamma_M(0)$, $q_n \rightarrow q_0 = 1/4 = \gamma_M(0)$, and $p_n \rightarrow p_0 = -2 = \gamma_M(1/2)$. Now if mating was continuous, we should have $R_0 \amalg Q_0 \simeq P_0 \amalg R_0$; both geometric matings exist, the former has two parabolic basins and the latter has one.
- For a parabolic or bounded-type Siegel parameter p on the boundary of the main cardioid with angle θ and the real parameter q defined by the Douady Magic Formula $\Theta = 1/2 + \theta/4$ according to Proposition ??, consider the geometric mating $f_\theta \cong P \amalg Q$, which exists according to Blé-Valdez [?, ?]. When θ is irrational, then $f_\theta^2(\infty) = 0$, since the corresponding point in \mathcal{K}_q has the angles $\pm 2\Theta = \pm\theta/2$ and the critical point of P has $\theta/2$ as well. But when θ is rational, then either 0 is in a parabolic basin and ∞ is preperiodic, or there are disjoint cycles of parabolic basins; in both cases $f_\theta^2(\infty) \neq 0$. So approximating a rational angle with irrational ones gives a contradiction to continuity.
- Theorem 8.1 below uses similar ideas to show that the limit is different from the expected one; since only rational angles are used, no special arguments are needed to show matability. Here both p_n and q_n are Misiurewicz polynomials; a concrete example is given below as well.
- Shared matings according to Theorem 6.2 can be used to produce several counterexamples to continuity; here p_n is hyperbolic and q_n is Misiurewicz. Again, the contradiction comes either from a different number of parabolic Fatou cycles, or from an expected limit outside of the Chebychev family.
- Different kinds of discontinuity may be expected in higher degrees. E.g., with cubic polynomials $f_a(z) = z^3 + az^2$, the mating $f_a \amalg f_{-\bar{a}}$ gives an antipode-preserving rational map [?]. The former bifurcation locus shall be locally connected at parabolic parameters, while the latter is not. So for suitable sequences of postcritically finite polynomials, there will be an oscillatory behavior.

Theorem 8.1 (Discontinuity with bitransitive family)

Tdiscbi

Consider a sequence of rational angles $\theta_n \rightarrow \theta_0$, such that θ_n and $2\theta_n$ are preperiodic for $n \geq 1$, $2\theta_0$ is periodic, and θ_0 may be either unless θ_0 and $2\theta_0$ belong to the same root. Set $p_n = \gamma_M(\theta_n)$ and $q_n = \gamma_M(-2\theta_n)$ for $n \geq 0$. Then the sequence of geometric matings $f_n \cong P_n \amalg Q_n$ does not converge to $f_0 \cong P_0 \amalg Q_0$.

Proof: First, note that θ and 2θ are never in the same limb, unless both are angles of the root. Thus all geometric matings under consideration exist. Since the angle θ_n of $p_n \in \mathcal{K}_{p_n}$ is complex conjugate to an angle $-\theta_n$ of $0 \in \mathcal{K}_{q_n}$, there is a direct ray connection between these two points, and the rational map satisfies $f_n(0) = \infty$. We have $f_n \not\rightarrow f_0$ since $f_0(0) \neq \infty$: while $z = \infty$ has an infinite orbit converging to a parabolic cycle of f_0 , $z = 0$ either has a finite orbit or it converges to a different parabolic cycle. — This phenomenon seems to be analogous to parabolic

implosion, if we are looking at the polynomials Q_n or at precaptures according to Proposition 4.1: $q_n = \gamma_{q_n}(-2\theta_n)$ converges to the critical value q_0 inside a parabolic Fatou component of Q_0 , but $\gamma_{q_0}(-2\theta_0)$ is a boundary point of this component. Of course, parabolic implosion looks different for the rational maps here, since the Julia set of f_n is all of $\widehat{\mathbb{C}}$. ■

A concrete example is given by $\theta_n = u_n/2^{2n}$ with $u_n = (2^{2n-1} + 1)/3$. Then p_n and q_n are β -type Misiurewicz points, converging to the Misiurewicz point $p_0 = i = \gamma_M(1/6)$ and the root $q_0 = -3/4 = \gamma_M(1/3)$, respectively, and the matings do not converge to the mating of the limits. Probably we have a parabolic 2-cycle in both cases, and Fatou components corresponding to a fat Basilica, but the limit of the matings has 0 and ∞ in different components of the Fatou set, while the mating of the limits has 0 in the Julia set at a preimage of the parabolic fixed point.

The program Mandel provides several interactive features related to the Thurston Algorithm. It is available from www.mndynamics.com. A console-based implementation of slow mating is distributed with the [preprint](#) of [?].