

# The Thurston Algorithm for quadratic matings

Wolf Jung

Gesamtschule Brand, 52078 Aachen, Germany,  
and Jacobs University, 28759 Bremen, Germany.

E-mail: [jung@mndynamics.com](mailto:jung@mndynamics.com)

*Dedicated to the memory of Tan Lei*

## Abstract

Mating is an operation to construct a rational map  $f$  from two polynomials, which are not in conjugate limbs of the Mandelbrot set. When the Thurston Algorithm for the unmodified formal mating is iterated in the case of postcritical identifications, it will diverge to the boundary of Teichmüller space, because marked points collide. Here it is shown that the colliding points converge to postcritical points of  $f$ , and the associated sequence of rational maps converges to  $f$  as well, unless the orbifold of  $f$  is of type  $(2, 2, 2, 2)$ . So to compute  $f$ , it is not necessary to encode the topology of postcritical ray-equivalence classes for the modified mating, but it is enough to implement the pullback map for the formal mating. The proof combines the Selinger extension to augmented Teichmüller space with local estimates.

Moreover, the Thurston Algorithm is implemented by pulling back a path in moduli space. This approach is due to Bartholdi–Nekrashevych in relation to one-dimensional moduli space maps, and to Buff–Chéritat for slow mating. Here it is shown that slow mating is equivalent to the Thurston Algorithm for the formal mating. An initialization of the path is obtained for repelling-preperiodic captures as well, which provide an alternative construction of matings.

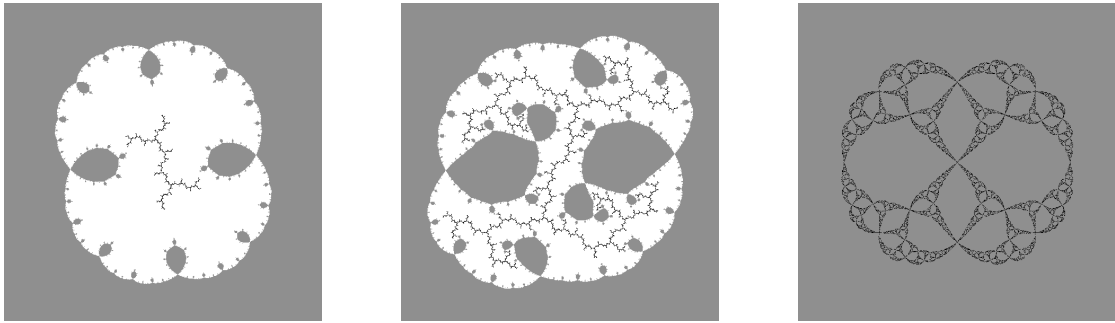
2010 MSC: 37F30 (Primary); 37F10, 37F45 (Secondary)

## 1 Introduction

A postcritically finite quadratic polynomial  $f_c(z) = z^2 + c$  may be periodic of satellite type, periodic of primitive type, or critically preperiodic (Misiurewicz type). Examples are given by the Basilica  $Q(z) = z^2 - 1$ , the Kokopelli, and by  $P(z) = z^2 + i$  in Figures 1 and 3. Quadratic rational maps have two critical orbits and form a two-parameter family. The dynamics and topology of certain rational maps is understood in terms of one or two polynomials [45]. The operation of mating was introduced by Douady–Hubbard [13]: a rational map  $f$  may be described by gluing the Julia sets of  $P$  and  $Q$ , such that points with conjugate external angles are identified. According to Rees–Shishikura–Tan [54, 52], this construction works when

$P$  and  $Q$  are not in conjugate limbs of the Mandelbrot set: first define the formal mating, where the two Julia sets are in separate half-spheres. The Thurston Theorem [14, 23] shows that there is an equivalent rational map  $f$ . Then the topological mating is given by collapsing all ray-equivalence classes of the formal mating, and it is conjugate to the geometric mating  $f$ . Actually, an intermediate step is required when postcritical points are identified in the mating: then the formal mating will be obstructed, and an unobstructed essential mating is constructed by collapsing a finite number of ray-equivalence classes.

The Thurston Algorithm is based on an iteration in Teichmüller space, which consists of isotopy classes of homeomorphisms, These may be represented by spiders, medusas, or triangulations. Bartholdi–Nekrashevych [2] and Buff–Chéritat [10] employ a path in moduli space instead. Using this “slow” approach, the algorithm shall be faster, easier to implement, and more stable. Explicit initializations are discussed in Sections 2.3, 5, and 6.1. The slow mating algorithm is related to equipotential gluing in [11]. Figure 1 shows a few snapshots of this process.



**Figure 1:** Various stages of slow mating, illustrated by moving images  $\psi_t(\varphi_0(\mathcal{K}_p))$  and  $\psi_t(\varphi_\infty(\mathcal{K}_q))$  of Julia sets. Here  $P(z) = z^2 + i$  is a Misiurewicz polynomial and  $Q(z) = z^2 - 1$  is the Basilica polynomial, which has an attracting 2-cycle. The formulas for pulling back marked points and rational maps are discussed in (3) and in Example 2.5 as well.

In this example, the Thurston Algorithm does not work directly, because the postcritical 2-cycle of  $P$  needs to be identified with a fixed point of  $Q$ : these are connected by external rays, and the ray-equivalence class is surrounded by a removable Thurston obstruction. The classical approach is to construct an essential mating, where certain ray-equivalence classes are collapsed by definition, and to employ the Thurston Algorithm for the modified map. An alternative approach is suggested here: the divergence of the Thurston Algorithm has been described by Nikita Selinger [48, 49] in terms of the augmented Teichmüller space. Applying his characterization to the Thurston Algorithm of the unmodified formal mating, it is shown that marked points come together automatically in the expected way, and the rational maps converge to the geometric mating, at least if the orbifold is not of type  $(2, 2, 2, 2)$ . The same argument gives convergence of slow mating and equipotential gluing as well, where no modification is appropriate. Thus it is possible to obtain matings numerically without encoding the topology of ray-equivalence classes. In a few more applications, additional obstructions are created and used to prove convergence properties [27, 28]. Here obstructions do not appear as a potential problem, but they are turned into an ally: a powerful tool to show convergence.

The classical Thurston Theorems are discussed in Section 2, together with the implementation by a path in moduli space. See Section 3 for examples of canonical obstructions and stabilization of noded Riemann surfaces, including a relation between core entropy and matings of conjugate polynomials. A general convergence result is obtained for bicritical maps in a suitable normalization. The various concepts of mating are developed systematically in Section 4. Convergence of mating and the implementation of slow mating and of captures is discussed in Sections 4.3, 5 and 6.1, respectively. Various algorithms are compared briefly in Section 7.

**Acknowledgment:** Several colleagues have contributed to this work by inspiring discussions and helpful suggestions. I wish to thank in particular Laurent Bartholdi, Walter Bergweiler, Xavier Buff, Arnaud Chéritat, Dzmitry Dudko, Adam Epstein, Sarah Koch, Michael Mertens, Carsten Petersen. Mary Rees, Pascale Roesch, Dierk Schleicher, Nikita Selinger, Tan Lei, Dylan P. Thurston, and Vladlen Timorin. And I am grateful to the mathematics department of Toulouse University Paul Sabatier for their hospitality.

*When I started to learn about complex dynamics some twenty years ago, the most impressive phenomenon was the self-similarity of the Mandelbrot set at Misiurewicz points, and Tan Lei's name is firmly attached to this. She has worked on parabolic maps, quasi-conformal deformations, and on vector fields as well. A recurring theme in her work is the Thurston characterization of rational maps, in particular its application to matings. I remember joyful discussions in Holbæk 2007, and in recent years we have had a few conversations about topological entropy. Tan Lei passed away in April 2016. This paper is dedicated to her memory.*

## 2 The Thurston Algorithm

The Thurston Theorem 2.7 gives a combinatorial characterization of branched covers equivalent to rational maps, which is used to describe and to define rational maps, and to construct them numerically: the related Thurston Algorithm provides a convergent sequence of rational maps. An underlying iteration in Teichmüller space  $\mathcal{T}$  is needed both to define a unique pullback, and to have analytic tools providing global convergence to a unique fixed point in  $\mathcal{T}$ . This fails in the presence of Thurston obstructions: then certain annuli get big, curves get short, marked points collide. This process is understood by extending the pullback map to augmented Teichmüller space  $\widehat{\mathcal{T}}$ ; see Section 3. The present section provides an introduction to the classical theory by William P. Thurston, and Section 2.3 discusses the implementation of the Thurston Algorithm in terms of a path in moduli space  $\mathcal{M}$ .

### 2.1 Hyperbolic geometry and Teichmüller spaces

A hyperbolic Riemann surface of finite type and genus 0 is isomorphic to the Riemann sphere  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  with  $n \geq 3$  punctures. Although the manifold extends analytically to a puncture or **marked point**, the hyperbolic metric is infinite there. We shall deal with homotopy classes of simple closed curves and the hyperbolic length of geodesics:

- A simple closed curve in the complement of the marked points is **essential**, if each disk in the complement of the curve contains at least two marked points;
- **peripheral**, if one component contains only one marked point; and
- **trivial** or null-homotopic, if one component contains no marked point.

Note that some authors say non-peripheral instead of essential or inessential instead of peripheral. The following properties of hyperbolic geodesics are fundamental:

**Proposition 2.1 (Hyperbolic geodesics)**

Consider the hyperbolic metric on  $\widehat{\mathbb{C}}$  with  $n \geq 3$  punctures:

1. For any essential simple closed curve there is a unique geodesic homotopic to it.
2. A simple closed geodesic  $\gamma$  has a **collar** neighborhood, an embedded annulus of definite width. Collars around disjoint geodesics are disjoint, and a geodesic crossing the collar of  $\gamma$  has an explicit lower bound on its length, which goes to  $\infty$  when  $l(\gamma) \rightarrow 0$ . In particular, all sufficiently short geodesics are disjoint.
4. Any annulus around  $\gamma$  has modulus bounded above by  $\pi/l(\gamma)$ . The collar has modulus bounded below by  $\pi/l(\gamma) - 1$ .
4. For a sequence of surfaces in a suitable normalization, two marked points collide with respect to the spherical metric, if and only if a hyperbolic geodesic separating them from two other marked points has length going to 0.

References for the **proof**: See [22] for item 1 and [14, 22, 9] for items 2 and 3. Item 4 is a standard estimate for extremal annuli. ■

For an implicit  $n \geq 3$ , **moduli space**  $\mathcal{M}$  is the space of Riemann spheres with marked points up to Möbius maps or normalization of three points; in our case of genus 0, it has an explicit description as a subset of  $\widehat{\mathbb{C}}^{n-3}$  from the positions of marked points. Now **Teichmüller space**  $\mathcal{T}$  is the universal cover of  $\mathcal{M}$ . It can be described by isotopy classes of orientation-preserving homeomorphisms  $\psi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  between spheres with marked points; here the left sphere is a topological sphere fixed for reference. Although it has no complex structure, let us write  $\widehat{\mathbb{C}}$  instead of  $S^2$  nevertheless: this allows to use explicit coordinates and formulas from  $\mathbb{C}$ . The **projection**  $\pi : \mathcal{T} \rightarrow \mathcal{M}$  gives the universal cover, and the **pure mapping class group**  $G$  is the group of deck transformations:  $[h] \in G$  is an isotopy class of homeomorphisms of the topological sphere fixing the marked points, which acts on  $\mathcal{T}$  by  $[h] \cdot [\psi] = [\psi \circ h^{-1}]$ ;  $G$  is generated by Dehn twists [22].

There are various metrics on  $\mathcal{T}$ , such that  $G$  acts by isometries and the metrics project to  $\mathcal{M}$ . Actually the definition as Finsler metrics is lifted from  $\mathcal{M}$  to  $\mathcal{T}$  locally. The dual tangent space is given by integrable quadratic differentials, and by using different norms there, the infinitesimal metrics  $\|d\tau\|_{\mathcal{T}}$  and  $\|d\tau\|_{WP}$  are obtained; integration along shortest curves defines the **Teichmüller metric**  $d_{\mathcal{T}}$  and the **Weil–Peterson metric**  $d_{WP}$  [22, 60, 33]. The Teichmüller metric is given equivalently by  $d_{\mathcal{T}}([\psi_1], [\psi_2]) = \inf 1/2 \log K(\psi)$ , where  $\psi$  is isotopic to  $\psi_1 \circ \psi_2^{-1}$  and  $K$  denotes the quasi-conformal dilatation.

**Proposition 2.2 (Basic properties of Teichmüller space)**

1.  $\mathcal{T}$  and  $\mathcal{M}$  are analytic manifolds and the universal cover  $\pi : \mathcal{T} \rightarrow \mathcal{M}$  is analytic. It is a local isometry for both metrics,  $d_{\mathcal{T}}$  and  $d_{WP}$ , and  $G$  acts by isometries.

2. Both metrics generate the same topology on  $\mathcal{T}$ . Now  $\mathcal{T}$  is complete and not compact with respect to  $d_{\mathcal{T}}$ , incomplete with respect to  $d_{WP}$ .
3. For an essential curve  $\gamma$  in the topological sphere and  $\tau \in \mathcal{T}$ , denote by  $l(\gamma, \tau)$  the **length** of the geodesic in the Riemann surface  $\pi(\tau)$ , that is homotopic to  $\psi(\gamma)$  for  $\psi \in \tau$ . This length is continuous on  $\mathcal{T}$  with  $|\log l(\gamma, \tau') - \log l(\gamma, \tau)| \leq 2d_{\mathcal{T}}(\tau', \tau)$ .
4. There are finitely many essential curves  $\gamma_i$ , such that the collection of length functions  $l(\gamma_i, \tau')$  determines  $\tau$  uniquely.
5. There is a relative estimate  $\|d\tau\|_{WP} \leq C_* \|d\tau\|_{\mathcal{T}}$  with  $C_* = C_*(\mathcal{T})$ .
6. For  $R > 0$  there is  $D_* = D_*(R, \mathcal{T}) > 0$  such that all  $\tau$  with shortest geodesic length  $l_*(\tau) \geq R$  satisfy: all  $\tau'$  with  $d_{WP}(\tau', \tau) \leq D_*$  have  $d_{\mathcal{T}}(\tau', \tau) \leq 1/4$ .
7. A closed subset of  $\mathcal{M}$  is compact, if and only if the length of all simple closed geodesics is bounded uniformly below.

References for the **proof**: See [22, 60, 33] for items 1–4 and [35] for item 5.

6. According to Lemma 3.22 in [33] we have the relative estimate

$$\|d\tau\|_{\mathcal{T}} \leq \frac{C}{l_*(\tau)} \|d\tau\|_{WP}, \quad (1)$$

where  $C$  depends only on the Teichmüller space  $\mathcal{T}$  and  $l_*$  denotes the length of the shortest hyperbolic geodesic on the Riemann surface  $\pi(\tau)$ . So we must choose  $D_*$  small enough to have a lower bound for  $l_*(\cdot)$  on the  $WP$ -geodesic from  $\tau'$  to  $\tau$ . Note that it is not sufficient to have a lower bound at  $\tau'$  and  $\tau$  only; cf. Remark 3.4. Suppose  $\tau'$  has  $l_*(\tau') < R/2$ , then for  $\varepsilon > 0$  there is a subarc  $[\tilde{\tau}', \tilde{\tau}]$  of the  $WP$ -geodesic  $[\tau', \tau]$  and a simple closed curve  $\gamma$ , such that  $l(\gamma, \cdot)$  increases from  $R/2 + \varepsilon$  to  $R$  along this subarc while  $l_*(\cdot) \geq R/2$ . Now  $|dl| \leq 2l \|d\tau\|_{\mathcal{T}}$  and (1) give

$$d_{WP}(\tau', \tau) > d_{WP}(\tilde{\tau}', \tilde{\tau}) \geq \frac{1}{2C} \int_{[\tilde{\tau}', \tilde{\tau}]} \frac{l_*(\cdot)}{l(\gamma, \cdot)} dl(\gamma, \cdot) \geq \frac{1}{4C} \int_{R/2+\varepsilon}^R dl, \quad (2)$$

so if  $D_* = \frac{R}{8C}$  and  $d_{WP}(\tau', \tau) \leq D_*$  then  $l_*(\cdot) \geq R/2$  on the  $WP$ -geodesic  $[\tau', \tau]$  and (1) gives  $d_{\mathcal{T}}(\tau', \tau) \leq \frac{2C}{R} \cdot D_* = 1/4$ .

7. This is the Mumford compactness theorem, whose proof is simplified in genus 0: the length is bounded below on any bounded ball. Every sequence in  $\mathcal{M}$  has a convergent subsequence in  $\widehat{\mathbb{C}}^{n-3}$ , but if the length of geodesics may go to 0, marked points collide according to Proposition 2.1.4 and the limit does not belong to  $\mathcal{M}$ . Note that every compact subset of  $\mathcal{T}$  has length bounded below as well, but the converse is wrong: for general  $g \in G$  the sequence  $\tau_k = g^k \cdot \tau_0$  does not accumulate in  $\mathcal{T}$ , but all  $l(\gamma, \tau_k)$  are independent of  $k$ . ■

## 2.2 Thurston maps and pullback map

A postcritically finite rational map  $f$  is not characterized uniquely by its ramification portrait. Except for flexible Lattès maps, the additional topological information can be given combinatorially. For polynomials, Hubbard trees and external angles provide an explicit description. For rational maps, the combinatorial object is an equivalence class of Thurston maps:

- A **Thurston map**  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is an orientation-preserving branched cover of degree  $d \geq 2$  with finite postcritical set  $P$  and marked set  $Z \supset P$ . Here  $P$  contains all forward iterates of critical points and  $Z$  may contain additional critical, preperiodic, and periodic points, such that  $g(Z) \subset Z$ .
- Two Thurston maps  $f, g$  are Thurston equivalent or **combinatorially equivalent**, if there are homeomorphisms  $\psi_0, \psi_1$  with  $\psi_0 \circ g = f \circ \psi_1$ ,  $\psi_0 = \psi_1$  on  $Z_g$ ,  $\psi_i(Z_g) = Z_f$ , and  $\psi_1$  is isotopic to  $\psi_0$  relative to  $Z_g$ . So  $g$  is isotopic to  $\psi_1^{-1} \circ f \circ \psi_1$ .
- A pullback map is associated with each Thurston map  $g$  as follows: for any homeomorphism  $\psi$  there is a rational map  $f$  and another homeomorphism  $\psi'$  with  $\psi \circ g = f \circ \psi'$ ; see [14, 23, 9]. The complex structure defined by  $\psi$  is pulled back with  $g$  and integrated with  $\psi'$ . These functions are unique up to Möbius maps, or unique after normalizing three marked points to  $\infty, 0, 1$ . It turns out that the isotopy class of  $\psi$  determines the isotopy class of  $\psi'$ , so an analytic **pullback map**  $\sigma_g : \mathcal{T} \rightarrow \mathcal{T}$  is defined by  $\sigma_g([\psi]) = [\psi']$ .
- The fixed points of  $\sigma_g$  in  $\mathcal{T}$  correspond to Möbius conjugacy classes of rational maps. Under suitable conditions, the pullback map will be strictly contracting, and the **Thurston Algorithm** converges in addition: define  $\psi_n \circ g = f_n \circ \psi_{n+1}$  recursively, then  $f_n \rightarrow f$  and  $[\psi_n]$  converges in  $\mathcal{T}$  to the fixed point of  $\sigma_g$ .

The ramification portrait of  $g$  translates to relations between the images of marked points,  $x_i = \psi(z_i)$  and  $x'_i = \psi'(z_i)$ , such that  $g(z_i) = z_j$  implies  $f(x'_i) = x_j$  when  $\psi \circ g = f \circ \psi'$ . In the bicritical case with marked critical points,  $f$  is determined by  $x_i$  and the  $x'_i$  are determined up to the branch of the  $d$ -th root. For the formal mating  $P \sqcup Q$  with  $P(z) = z^2 + i$  and  $Q(z) = z^2 - 1$  according to Figure 1 and Example 2.5, this reads

$$x'_1 = \pm \sqrt{\frac{x_1 - x_2}{1 - x_2}} \quad x'_2 = \pm \sqrt{\frac{2x_1}{1 + x_1}}. \quad (3)$$

Note that we cannot pull back marked points with these formulas alone, since the choice of branch is determined by the pullback in Teichmüller space. The following proposition gives contraction properties of  $\sigma_g$ ; see Section 2.4 for the notation  $(2, 2, 2, 2)$  and for the relation between convergence and Thurston obstructions.

**Proposition 2.3 (Thurston–Selinger)**

Consider a Thurston map  $g$  of degree  $d \geq 2$  and the pullback map  $\sigma_g$ .

1.  $\sigma_g$  is weakly contracting with respect to the Teichmüller metric.
2. If  $g$  has orbifold type not  $(2, 2, 2, 2)$ , then some iterate of  $\sigma_g$  is strictly contracting. The contraction is uniform on subsets of  $\mathcal{T}$ , such that  $\pi(\tau)$  varies in a compact subset of  $\mathcal{M}$ .
3.  $\sigma_g$  is Lipschitz continuous on  $\mathcal{T}$  with respect to the Weil-Petersson metric; a factor is given by  $\sqrt{d}$ .

References for the **proof**: 1: Weak contraction follows from the definition in terms of a minimal dilation, or since the Teichmüller metric is a Kobayashi metric [22].

2. The contraction with respect to the infinitesimal Teichmüller metric is obtained from the dual operator, a push-forward of quadratic differentials. An explicit integral gives strict contraction, unless there is a specific type  $(2, 2, 2, 2)$  of branch portrait. See [14, 23], and [9] for the case of additional marked points. Uniform contraction follows from the fact that the rational maps  $f$  depend only on a finite intermediate cover of  $\mathcal{M}$ ; cf. Section 3.4.
3. Apply the Cauchy–Schwartz inequality to the same integral representation of the push-forward operator, to obtain an estimate in the  $L^2$ -norm [48]. ■

### 2.3 A path in moduli space

The pullback of homeomorphisms  $\psi_n$  was easy to define, but it is not computed easily: repeated pullbacks would be defined piecewise, and solving the Beltrami equation numerically would be impractical as well. The isotopy classes in Teichmüller space are meant to represent only combinatorial information anyway: we are interested in the pullback of marked points  $x_i(n) \in \pi(\sigma_g^n([\psi_0]))$  and maps  $f_n$ , and the combinatorial description is needed to make a finite choice between different possible preimages. This characterization of the topology has been implemented in terms of spiders [5, 20], medusas [7], and triangulations [3]. These contain the necessary information from Teichmüller space without being actual homeomorphisms  $\psi_n$ .

Following Bartholdi–Nekrashevych [2] and Buff–Chéritat [10], the following alternative method shall be discussed. It means that Teichmüller space is used explicitly only to check a suitable initialization of a path in moduli space. Afterward the path is pulled back simply by choosing preimages from continuity. The application to matings is discussed in Sections 5 and 6. A few more applications to rational maps are given in [28]. The spider algorithm is implemented with a path in [29] and further applications to quadratic polynomials are given; twisted polynomials and Lattès maps are discussed in [27] as well.

#### Proposition 2.4 (Path in moduli space)

*Suppose  $g$  is a Thurston map of degree  $d \geq 2$ , and there is a continuous path of homeomorphisms  $\psi_t : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ ,  $0 \leq t \leq 1$ , with  $\psi_0 \circ g = f_0 \circ \psi_1$  for a rational map  $f_0$ . So  $[\psi_1] = \sigma_g([\psi_0])$ .*

1. *Using a suitable normalization, there is a unique path of homeomorphisms  $\psi_t$ ,  $0 \leq t < \infty$ , with  $\psi_t \circ g = f_t \circ \psi_{t+1}$  for rational maps  $f_t$ , so  $[\psi_{t+1}] = \sigma_g([\psi_t])$ . It projects to a continuous path  $\pi([\psi_t])$  in moduli space. Note that  $\sigma_g^n([\psi_0]) = [\psi_n]$  for  $n \in \mathbb{N}$ .*

2. *Suppose that  $d = 2$ , or more generally, that  $g$  is bicritical. Normalize the marked points  $x_i(t) \in \pi([\psi_t])$  such that 0 and  $\infty$  are critical and 1 is postcritical or marked in addition. Then the path  $x_i(t)$  in moduli space is computed for  $1 \leq t < \infty$  by pulling back the initial segment continuously.*

Probably the statement remains true when  $g$  is not bicritical, but the pullback is less explicit, and I am not sure if it is unique in general. Note that  $[\psi_1] = \sigma_g([\psi_0])$  and an initial path  $\psi_t$  is projected to moduli space. If this condition is neglected by choosing an arbitrary path from  $\pi([\psi_0])$  to  $\pi([\psi_1])$ , the pullback may correspond not to  $g$  but to some twisted version of it. Conditions for convergence of  $\sigma_g^n([\psi_0])$

are discussed in Section 2.4; in the case of a non-(2, 2, 2, 2) orbifold, convergence in Teichmüller space is equivalent to convergence in moduli space, and in both spaces, convergence of the sequence implies convergence of the path as  $t \rightarrow \infty$ . The situation is more involved for an orbifold of type (2, 2, 2, 2). The implementation in terms of a piecewise linear path is discussed in [28, 29].

**Proof:** 1.  $\sigma_g$  and  $\pi$  are continuous. Marked points never meet under iterated pullback, so  $\psi_{t+1}$  is always defined uniquely up to Möbius conjugation.

2. In this normalization, we have  $f_t(z) = m_t(z^d)$ , and the Möbius transformation  $m_t$  is determined uniquely from the images of 0, 1,  $\infty$  at time  $t$ . The path is pulled back uniquely by  $f_t^{-1}(z) = \sqrt[d]{m_t^{-1}(z)}$ , since any coordinate is either constant 0 or  $\infty$ , or the argument of the radical is never passing through 0 or  $\infty$ . ■

### Example 2.5 (Misiurewicz polynomial mates Basilica)

The mating of the Misiurewicz polynomial  $P(z) = z^2 + i$  and the Basilica polynomial  $Q(z) = z^2 - 1$  is illustrated in Figure 1. Consider the Thurston Algorithm for the formal mating  $g$  with a path according to Initialization 5.1 and the radius  $R_t = \exp(2^{1-t})$ . Rescaled to  $f_t(\infty) = 1$ , the initialization for  $0 \leq t \leq 1$  reads

$$x_1(t) = -i/R_t^2 \quad x_2(t) = \frac{(1-i)/R_t^2}{1+(1-t)e^{-4}} \quad x_3(t) = \frac{i/R_t^2}{1+(1-t)2ie^{-4}}. \quad (4)$$

Note that the normalization  $x_3(t) = -x_1(t)$  is satisfied for  $t \geq 1$  only. For  $t \geq 0$  we have the following pullback relation, and the formula for  $x_2(t+1)$  simplifies to (3) when  $t \geq 1$ :

$$x_1(t+1) = \pm \sqrt{\frac{x_1(t) - x_2(t)}{1 - x_2(t)}} \quad x_2(t+1) = \pm \sqrt{\frac{x_1(t) - x_3(t)}{1 - x_3(t)}} \quad x_3(t+1) = -x_1(t+1), \quad (5)$$

where the sign is chosen by continuity. According to Theorem 4.3, the rational maps  $f_t$  converge to the rescaled geometric mating  $f(z) = (z^2 + 2)/(z^2 - 1)$ , so  $x_1(t) \rightarrow -2$ ,  $x_2(t) \rightarrow 2$ , and  $x_3(t) \rightarrow 2$ . Since two postcritical points are identified, the iteration diverges in moduli space and in Teichmüller space.

An alternative interpretation of the path reads as follows: by a standard technique from algebraic topology, the universal cover of moduli space is constructed as the space of homotopy classes of paths with a fixed starting point. So that space is isomorphic to Teichmüller space. In this sense, the pullback of the path is a direct implementation of  $\sigma_g$ , and information on the dynamics of  $\sigma_g$  is available from homotopy classes of paths. See Section 3.3 in [27] for an application.

Sarah Koch [30] gives criteria on  $g$  for the existence of a moduli space map from  $\pi(\sigma_g([\psi]))$  to  $\pi([\psi])$ , which is a critically finite map in the same dimension as the moduli space. See also Section 3.2 in [27]. Then the path may be chosen within the Julia set of the moduli space map, which is easily visualized when it is one-dimensional [2]. This happens for a NET map, which has four postcritical points and only simple critical points [15]. In the quadratic case of NET maps, a moduli space map exists if at least one critical point is postcritical, and not when  $g$  is a Lattès map of type (2, 2, 2, 2).



### Example 2.6 (Obstructed self-mating)

For the self-mating of the Basilica polynomial  $P(z) = Q(z) = z^2 - 1$ , consider the radius  $R_t = \exp(2^{1-t})$  again, and Initialization 5.1 reads  $x_1(t) = -1/R_t$  for  $0 \leq t \leq 1$ . The normalization is symmetric under inversion, and the pullback relation  $x_1(t+1) = -\sqrt{-x_1(t)}$  has an explicit solution in this case, which is given by  $x_1(t) = -1/R_t$  for  $0 \leq t < \infty$ . So  $x_1(t) \rightarrow -1$  as  $t \rightarrow \infty$ , and the rational maps  $f_t(z) = (z^2 + x_1(t))/(1 + x_1(t)z^2)$  degenerate to a constant map. Note that there is a moduli space map  $x_1(t) = -(x_1(t+1))^2$ , and for a different initialization, the path would be contained in the unit circle.

## 2.4 Obstructions and the Thurston Theorems

A **multicurve** is a nonempty union of pairwise disjoint and non-homotopic essential curves, or a homotopy class as well. For a concrete multicurve  $\Gamma$ , the preimage under a Thurston map  $g$  is  $\Gamma' \cup \Gamma''$ , where  $\Gamma'$  consists of essential curves and the curves in  $\Gamma''$  are peripheral or trivial. The homotopy class of  $\Gamma'$  depends only on the homotopy classes of  $\Gamma$  and  $g$ . Note that  $\Gamma'$  may contain mutually homotopic curves and it may be empty as well.  $\Gamma$  is called **invariant**, if every curve in  $\Gamma'$  is homotopic to a curve in  $\Gamma$ ; it is **completely invariant** if the converse holds in addition.

For  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$  the Thurston matrix  $M_\Gamma = (m_{ij})$  is defined as  $m_{ij} = \sum 1/d_{ijk}$ , where the sum runs over all preimages of  $\gamma_j$  homotopic to  $\gamma_i$  and  $d_{ijk}$  is the degree of  $g$  on these preimages. Now  $\Gamma$  is a Thurston **obstruction**, if the leading eigenvalue  $\lambda_\Gamma$  of  $M_\Gamma$  satisfies  $\lambda_\Gamma \geq 1$ . There are different conventions, whether invariance is required. Most important are the following kinds of obstructions:

- An invariant multicurve  $\Gamma$  is a **simple obstruction**, if no permutation turns  $M_\Gamma$  into a lower-triangular block form, such that the upper left block has leading eigenvalue  $< 1$ . A simple obstruction is always completely invariant.
- A multicurve  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$  is a **Lévy cycle**, if each  $\gamma_i$  is homotopic to a preimage of  $\gamma_{i+1 \bmod n}$  and the corresponding degree is one. Then  $\Gamma$  need not be invariant, but it can be extended to a simple obstruction. The converse holds in the quadratic or bicritical case: every simple obstruction contains a Lévy cycle. These are classified further in [54, 45].

Obstructions are important for the Thurston pullback, because they are related to the presence of annuli with large modulus and of short geodesics [14, 23, 9]; see also Section 3.2 for more explicit statements. Note that by adding more additional marked points,  $g$  may have more obstructions. The notion of an orbifold is explained in [40, 34]. A Thurston map of orbifold type  $(2, 2, 2, 2)$  has four postcritical points, the critical points are non-degenerate and not postcritical.

### Theorem 2.7 (Thurston–Pilgrim, general case)

Suppose  $g$  is a Thurston map of degree  $d \geq 2$  with orbifold not of type  $(2, 2, 2, 2)$ , possibly with additional marked points. Then:

- Either there is no Thurston obstruction,  $g$  is combinatorially equivalent to a rational map  $f$ , which is unique up to Möbius conjugation, and the Thurston pullback  $\sigma_g$  converges globally to its unique fixed point.

- Or  $g$  is obstructed, it is not equivalent to a rational map, and the Thurston pullback diverges. There is a unique **canonical obstruction**  $\Gamma$ , such that every  $\gamma \in \Gamma$  is pinched, while the length of every other curve is bounded uniformly from below.

Note that the fine print reads as follows: if there is an obstruction, it need not be pinching, but then it implies the existence of another obstruction, which is pinching. The pinching obstructions imply that marked points get identified in the limit, which means divergence to the boundary in Teichmüller space and in moduli space as well. Then the rational maps may converge to a rational map of degree  $d$  or of lower degree. Note that there is no algorithm to find obstructions in a time bounded a priori.

**Idea of the proof:** According to Proposition 2.3.2, some iterate of  $\sigma_g$  is strictly contracting. So if  $g$  is equivalent to a rational map  $f$ ,  $\sigma_g$  has a unique and globally attracting fixed point. If  $g$  was obstructed, there would be a simple obstruction  $\Gamma$ . Choosing  $\tau_0$  with suitable annuli of large modulus, these would not shrink under the pullback, giving a contradiction to the smaller annuli for  $f$ . Note also that a rational map is obstructed only if it is of flexible Lattès type, or postcritically infinite with a rotation domain [34].

On the other hand, if  $\sigma_g$  does not have a fixed point, the pullback  $\tau_n = \sigma_g^n(\tau_0)$  has the property that  $\pi(\tau_n)$  leaves every compact subset of  $\mathcal{M}$ ; there will be short geodesics by Mumford compactness according to Proposition 2.2.7. Now a combinatorial analysis gives a simple obstruction from curves sufficiently short and shorter than other curves. See [14, 23], and [9] for the case of additional marked points. The proof was refined by Pilgrim [43] to show that the same curves either do get shorter or stay bounded. For an alternative proof by Selinger [48], see Section 3.2. ■

Originally this theorem was stated under the assumption of a hyperbolic orbifold [14, 23]. A parabolic orbifold is either of type  $(2, 2, 2, 2)$  as discussed below, or there are only two or three postcritical points. In that case, the Thurston pullback is undefined or constant, respectively, unless there are additional marked points. When there are additional marked points, the pullback map is strictly contracting as in the hyperbolic orbifold case [9].

For quadratic Thurston maps of type  $(2, 2, 2, 2)$ , some things are the same, some are different: a fixed point is still unique, when it exists, but it is not attracting. Every obstruction is pinching, and it excludes a fixed point, but there are unobstructed maps not equivalent to a rational map as well. The converse happens when the degree  $d \geq 4$  is a square: there is a one-parameter family of flexible Lattès maps, which are mutually equivalent but not Möbius conjugate. So uniqueness fails, and moreover, there is a non-pinching obstruction. For Thurston maps of type  $(2, 2, 2, 2)$  with additional marked points, pinching and non-pinching obstructions are characterized by Selinger–Yampolsky [49, 50].

### Theorem 2.8 (Thurston, exceptional case)

1. Suppose  $g$  is a Thurston map of degree  $d \geq 2$  with orbifold of type  $(2, 2, 2, 2)$ , without additional marked points. Then it is equivalent to a map covered by a real-affine map on a torus, which is described by an integer matrix of determinant  $d$ :

- If the eigenvalues are not real, there is an equivalent rational map  $f$ , unique up to Möbius conjugation. The Thurston pullback map  $\sigma_g$  is represented by a Möbius transformation of the upper halfplane. It has a neutral fixed point, so the Thurston

pullback does not converge.

- If the matrix is a multiple of the identity,  $g$  is equivalent to a family of flexible Lattès maps.

- Otherwise,  $g$  is not equivalent to a rational map.

2. Suppose  $g$  is quadratic of type  $(2, 2, 2, 2)$ , without additional marked points, and the integer matrix has trace  $t$ :

- If  $|t| \leq 2$ , then  $g$  is equivalent to a rational map  $f$ , unique up to Möbius conjugation, and  $\sigma_g$  has a neutral fixed point.

- If  $|t| = 3$ , then  $g$  has a pinching obstruction, and the Thurston pullback diverges to the boundary in Teichmüller space and in moduli space as well.

- If  $|t| \geq 4$ , then  $g$  is not equivalent to a rational map either, but there is no Thurston obstruction for  $g$ . The Thurston pullback diverges to the boundary in Teichmüller space, but it is bounded in moduli space.

**Sketch of the proof:** 1. The lift to an affine map is explained in [14, 23, 34, 50, 6], see also [27]. In the flexible Lattès case, the choice of lattice is arbitrary, and in the case of non-real eigenvalues, the lattice can be chosen such that the real-affine map is holomorphic. The pullback of a constant Beltrami coefficient is described explicitly by a Möbius transformation of the upper halfplane with integer coefficients; a fixed point cannot be attracting, since otherwise the complex conjugate fixed point would be repelling.

2. The eigenvalues are computed from  $\eta^2 - t\eta + 2 = 0$ . When  $|t| \geq 4$ , they are real and not integer, so there is no invariant multicurve according to [49]. Note also that any possible obstruction would be a single curve  $\gamma$  separating the critical values from the other two marked points. Its two preimages are mutually homotopic, so we have a pinching obstruction when they are homotopic to  $\gamma$  as well. This cannot happen when the two critical values are mapped to the same prefixed point, which is equivalent to  $|t|$  being even. See also the example of twisting the self-mating  $f \cong 1/4 \amalg 1/4$  in Section 3.3 of [27]. ■

— Dylan P. Thurston [55] gives a positive criterion for  $g$  to be equivalent to a rational map, at least if there is a periodic critical point:  $f^{-k}$  is uniformly contracting on a graph, which forms a spine for  $\widehat{\mathbb{C}} \setminus P$ . In [44], Kevin Pilgrim gives an algebraic characterization of obstructions by non-contraction of the virtual endomorphism of the pure mapping class group. See [2, 4] for algebraic descriptions of Thurston maps in terms of iterated monodromy groups or bisets.

### 3 Extension to the augmented Teichmüller space

When a Thurston map  $g$  is not combinatorially equivalent to a rational map, so  $\sigma_g$  has no fixed point in  $\mathcal{T}$ , we may understand this by considering a space larger than  $\mathcal{T}$  or a different topology. Except for orbifold type  $(2, 2, 2, 2)$ , divergence of the Thurston Algorithm is related to collisions of marked points and pinching of essential curves. These phenomena are described by strata of augmented Teichmüller space  $\widehat{\mathcal{T}}$  and augmented moduli space  $\widehat{\mathcal{M}}$ , which parametrize noded Riemann surfaces. Selinger [48, 49] has extended  $\sigma_g$  to  $\widehat{\mathcal{T}}$  and obtained a related characterization of the canonical obstruction  $\Gamma$ . — A somewhat informal introduction to augmented spaces

is given in Section 3.1. Results on extension and characterization are discussed in Sections 3.2–3.3, and applied to obtain a relation between core entropy and matings of conjugate polynomials. The main result of the present paper is a convergence principle for obstructed maps, which is proved in Sections 3.4–3.5.

*Special thanks to Nikita Selinger for patiently answering my impatient questions.*

### 3.1 Augmented Teichmüller space and moduli space

Augmented moduli space  $\widehat{\mathcal{M}}$  describes noded Riemann surfaces, and augmented Teichmüller space  $\widehat{\mathcal{T}}$  has continuous maps from a topological sphere to a noded sphere; these send certain curves to single points. Boundary strata of  $\widehat{\mathcal{T}}$  are products of lower-dimensional Teichmüller spaces. The notion of noded Riemann surfaces is motivated here by pinching obstructions; originally they were introduced to compactify  $\mathcal{M}$ , and to describe algebraic curves with self-intersections. These constructions are due to Deligne–Mumford, Bers, Abikoff, and Masur; see the references in [21, 60, 33]. The following example shows the degeneration of a Riemann surface with boundary explicitly:

**Example 3.1 (Pinching a short geodesic)**

*Consider the Riemann surface  $S_t = \{(x, y) \in \mathbb{C}^2 \mid |x| < 1, |y| < 1, x \cdot y = t\}$  for  $|t| < 1$ . As  $t \rightarrow 0$ ,  $S_t$  becomes the union of two disks intersecting transversely in the single point  $(0, 0)$ . The hyperbolic metric in the annulus  $|t| < |x| < 1$  is known explicitly, and seen to converge to the hyperbolic metric of the punctured disk. When we try to illustrate this process in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , either  $S_t$  looks disconnected or the limit does not show a transversal intersection of smooth manifolds.*

This example shall motivate that we are interested in surfaces consisting of smooth spheres intersecting transversely. The nodes appear as additional marked points in the pieces, because the hyperbolic metric is singular there. An approximate Riemann surface would have long, thin tunnels between thick components; this is symbolized by connected spheres as in Figure 2.

**Example 3.2 (Augmented moduli space)**

1. Consider  $\widehat{\mathbb{C}}$  with four marked points  $x_1 = \infty$ ,  $x_2 = 0$ ,  $x_3 = 1$ ,  $x_4 = a$ . The moduli space is given by  $a \in \widehat{\mathbb{C}} \setminus \{\infty, 0, 1\}$  and the augmented moduli space is  $\widehat{\mathcal{M}} = \widehat{\mathbb{C}}$ : e.g.,  $a \rightarrow 0$  corresponds to pinching a curve separating  $x_2 = 0$  and  $x_4 = a$  from  $x_3 = 1$  and  $x_1 = \infty$ . Note that division by  $a$  gives a different normalization  $x_1 = \infty$ ,  $x_2 = 0$ ,  $x_3 = 1/a$ ,  $x_4 = 1$ ; now  $a \rightarrow 0$  means  $x_3 \rightarrow x_1$ . In fact this is the same Riemann surface as before, with a node separating  $x_2, x_4$  from  $x_3, x_1$ .

2. The case of five marked points is described by  $\mathcal{M} = \{(a, b) \in \widehat{\mathbb{C}}^2 \mid a, b \neq \infty, 0, 1, a \neq b\}$ , but now the topology of  $\widehat{\mathcal{M}}$  is more involved than  $\widehat{\mathbb{C}}^2$ : E.g., one-dimensional boundary strata are given by  $a = 0, b \neq \infty, 0, 1$  or by  $a = b \neq \infty, 0, 1$ , but we lose information when  $a = b = 0$ : this may be one of three 0-dimensional strata, or in a one-dimensional stratum without information on the relative position of three marked points and a node.

So  $\widehat{\mathcal{T}}$  contains maps from a topological sphere to a possibly noded Riemann surface, sending certain curves to nodes, equivalent under an isotopy in the domain or Möbius

transformations in the range. Boundary strata  $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$  are labeled by homotopy classes of pinched multicurves. There are only finitely many boundary strata  $\mathcal{S}_{G,\Gamma} \subset \widehat{\mathcal{M}}$ ; as in the case of  $\mathcal{T}$  and  $\mathcal{M}$ , different curves in the topological sphere may be mapped to the same short geodesic. A neighborhood basis for the topology of  $\widehat{\mathcal{T}}$  or  $\widehat{\mathcal{M}}$  is defined in terms of maps between noded Riemann surfaces, which map  $\varepsilon$ -short geodesics to nodes and which are  $(1+\varepsilon)$ -quasiconformal outside of the collars. Using a combination of Fenchel–Nielsen coordinates [22] and plumbing coordinates, which take Example 3.1 as a local model, the infinitely branched covering  $\pi : \widehat{\mathcal{T}} \rightarrow \widehat{\mathcal{M}}$  is understood locally. So  $\widehat{\mathcal{M}}$  is a compact analytic space [21].

**Proposition 3.3 (Augmented Teichmüller space)**

$\widehat{\mathcal{T}}$  and  $\widehat{\mathcal{M}}$  are topological spaces, such that  $\pi : \mathcal{T} \rightarrow \mathcal{M}$  extends to a continuous map  $\pi : \widehat{\mathcal{T}} \rightarrow \widehat{\mathcal{M}}$ .

1. The Weil–Petersson metric  $d_{WP}$  extends to  $\widehat{\mathcal{T}}$  and  $\widehat{\mathcal{M}}$ , such that  $\widehat{\mathcal{T}}$  is the completion of  $\mathcal{T}$  and  $\widehat{\mathcal{M}}$  is a compactification of  $\mathcal{M}$ . On each boundary stratum, the extended  $d_{WP}$  is a product of lower-dimensional Weil–Petersson metrics.
2. Each point  $\tau \in \widehat{\mathcal{T}}$  is approximated only from finitely many strata.
3. Normalizing three marked points, the coordinates of all marked points extend continuously from  $\mathcal{M}$  to  $\widehat{\mathcal{M}}$  or from  $\mathcal{T}$  to  $\widehat{\mathcal{T}}$ .
4. For every essential simple closed curve  $\gamma$ , the length  $l(\gamma, \tau) \in [0, \infty]$  is continuous on  $\widehat{\mathcal{T}}$ . All length functions together determine  $\tau$  uniquely.

Explanations, references, and sketch of a **proof**: See [21, 60, 33] for item 1. Note that  $\widehat{\mathcal{T}}$  is only a partial compactification of  $\mathcal{T}$ : a sequence leaving  $\mathcal{T}$  may converge in  $\widehat{\mathcal{T}}$ , if closed curves are pinched, but a sequence of the form  $g^k \cdot \tau_0$  will diverge in  $\widehat{\mathcal{T}}$  as well.

2. This follows from the definition of neighborhoods given above.
3. Continuity is obtained from extending the  $(1+\varepsilon)$ -quasiconformal maps into approximately round collars, or from a compactness argument and continuity of length. The normalization singles out a sphere, where all marked points have limits, while marked points in other components converge to nodes of this sphere. The statement is equivalent to a continuous extension of cross-ratios; in [16] a completion with respect to cross-ratios is used to construct a space isomorphic to  $\widehat{\mathcal{T}}$ .
4. See the references above and [48]. Approaching a lower-dimensional stratum according to item 2, specific curves have length  $\rightarrow 0$  and intersecting curves have length  $\rightarrow \infty$ . For all other curves, the hyperbolic metric converges on each component in a suitable normalization. Injectivity of lengths follows from Proposition 2.2.4. ■

The pure mapping class group  $G$  acts on  $\widehat{\mathcal{T}}$  by Weil–Petersson isometries, but the description of  $\widehat{\mathcal{M}} = \widehat{\mathcal{T}}/G$  is more involved:

**Remark 3.4 (Action of  $G$  and uniqueness of geodesics)**

Near a boundary stratum  $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$ , distinguish the following kinds of Dehn twists  $g \in G$  about  $\gamma$ :

- a) If  $\gamma$  intersects a curve in  $\Gamma$ , the action of  $g$  would map to a different stratum.
- b) If  $\gamma$  is contained in a component of  $\widehat{\mathcal{C}} \setminus \Gamma$ , then  $g$  acts from the pure mapping class group of the component space.

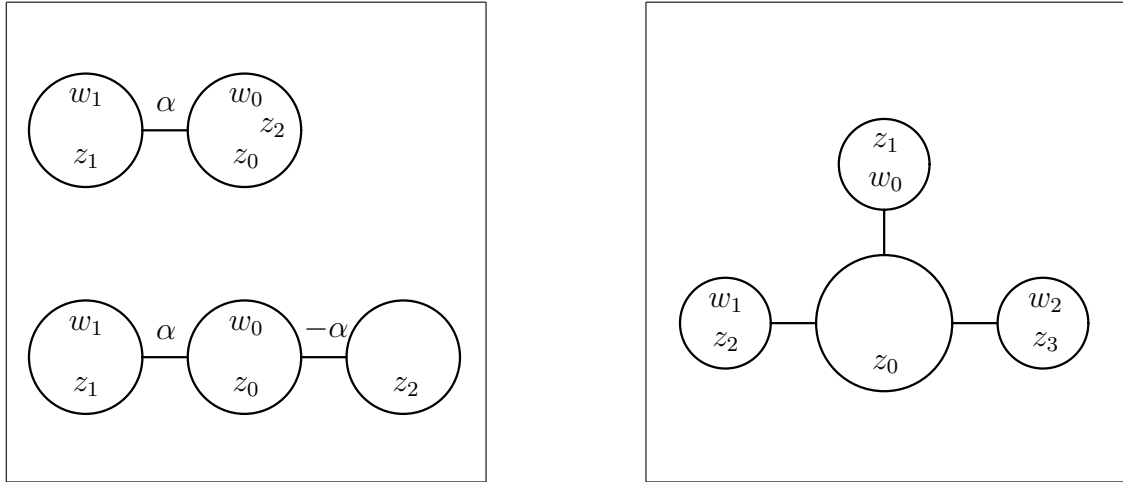
c) If  $\gamma \in \Gamma$ , then  $g$  acts trivially on the stratum but not trivially in a neighborhood. For  $\tau \in \mathcal{T}$  close to  $\mathcal{S}_\Gamma$ ,  $d_{WP}(\tau, g^k \cdot \tau)$  is bounded by the triangle inequality, but  $d_T(\tau, g^k \cdot \tau) \rightarrow \infty$ , although all hyperbolic geodesics are bounded below; cf. Proposition 2.2.

So  $\pi$  is infinitely branched and not a local isometry. According to [60],  $\widehat{\mathcal{T}}$  is a unique geodesic space nevertheless, with open geodesics passing through a unique stratum of lowest possible dimension.

### 3.2 Extended pullback map and the canonical obstruction

*Well, the extension to the augmented Teichmüller space exists independently of how or whether we understand it. — Nikita Selinger [private communication]*

To extend the pullback map  $\sigma_g$  to augmented Teichmüller space, consider a multi-curve  $\Gamma$  and the collection  $\Gamma'$  of non-homotopic essential preimages. Then  $\sigma_g$  shall map  $\mathcal{S}_\Gamma$  to  $\mathcal{S}_{\Gamma'}$ . The definition is understood by considering the full preimage  $g^{-1}(\Gamma)$  first; this defines noded surfaces with possibly non-hyperbolic pieces. So whenever a disk component contains at most one marked point, it is reduced to a point, and an annulus between homotopic curves is reduced to a point as well. This process of stabilization defines a noded Riemann surface with hyperbolic pieces.



**Figure 2:** Left: The formal mating  $g$  of Airplane and Basilica has a cyclic ray connection  $\Gamma = \{\gamma\}$  between the two  $\alpha$ -fixed points, which is the canonical obstruction. The preimage  $\Gamma' \cup \Gamma''$  contains a peripheral curve, so the right sphere in the lower surface  $\widehat{\mathcal{C}}/\Gamma'$  is considered as one point. Then  $g_\Gamma$  is a self-map of the noded surface  $\widehat{\mathcal{C}}/\Gamma$  and  $z_0$  is no longer 3-periodic  $z_0 \Rightarrow z_1 \rightarrow z_2 \rightarrow z_0$ , but preperiodic with  $z_0 \Rightarrow z_1 \rightarrow z_2 \rightarrow \alpha \uparrow$ .

Right: The formal mating  $1/4 \sqcup 1/2$  with marked critical points has three curves in the canonical obstruction, which surround ray connections with the angles  $1/4, 1/2, 0$ . Due to the identification of points in the small pieces, the geometric mating  $f \cong 1/4 \sqcup\sqcup 1/2$  satisfies  $f(0) = \infty$ .

So when  $\sigma_g([\psi]) = [\psi']$  and  $\psi$  maps the curves of  $\Gamma$  to nodes, then  $\psi'$  maps curves in  $\Gamma'$  to nodes, and certain annuli and disks to nodes or marked points as well. The pullback map on a product of lower-dimensional Teichmüller spaces is described in

terms of homeomorphisms or Thurston maps  $g^C$  on the pieces  $C$ , which are defined uniquely up to combinatorial equivalence. Note that for a completely invariant multicurve  $\Gamma$ , appropriate identifications must be made to describe a  $\sigma_g$ -invariant boundary stratum  $\mathcal{S}_\Gamma$ . The collection  $g_\Gamma$  of component maps is defined on the topological model surface  $\widehat{\mathbb{C}}/\Gamma$ . The examples in Figure 2 show that it may be discontinuous, not surjective, and it may map marked points to nodes.

**Theorem 3.5 (Selinger extension)**

*For a Thurston map  $g$  of degree  $d \geq 2$ , the Thurston pullback  $\sigma_g$  has a unique continuous extension to  $\widehat{\mathcal{T}}$ . On each boundary stratum, it is given by a pullback with component maps as described above.*

**Idea of the proof:** A unique extension is given by completion, using the uniform Lipschitz estimate from Proposition 2.3.3. For the explicit extension above, the length functions of all geodesics are continuous when a lower-dimensional stratum is approximated from a higher-dimensional stratum [48]. By Proposition 3.3.4, both extensions agree. ■

The following result is due to Pilgrim [43] in the case of a hyperbolic orbifold. The proof by Selinger [48] works for maps of type  $(2, 2, 2, 2)$  as well.

**Theorem 3.6 (Canonical obstruction by Pilgrim–Selinger)**

*Suppose  $g$  is a Thurston map of degree  $d \geq 2$ , fix  $\tau_0 \in \mathcal{T}$ , and set  $\tau_n = \sigma_g^n(\tau_0)$ . There is an  $R(\tau_0) > 0$  and a multicurve  $\Gamma$ , possibly empty, such that:*

- *If  $\gamma \in \Gamma$ , then  $l(\gamma, \tau_n) \rightarrow 0$ .*
- *If  $\gamma \notin \Gamma$ , then  $l(\gamma, \tau_n) \geq R(\tau_0)$ .*

*In augmented moduli space,  $\pi(\tau_n)$  accumulates at a compact subset of the canonical stratum  $\mathcal{S}_{G,\Gamma} \subset \widehat{\mathcal{M}}$ . The **canonical obstruction**  $\Gamma$  is independent of  $\tau_0$ . If  $\Gamma \neq \emptyset$ , it is a simple Thurston obstruction, and the curves of  $\Gamma$  do not intersect another curve from any simple obstruction.*

This implies that every accumulation point of  $\tau_n$  belongs to the canonical stratum  $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$ , but there need not be accumulation in  $\widehat{\mathcal{T}}$  at all. The accumulation sets in  $\widehat{\mathcal{M}}$  and  $\widehat{\mathcal{T}}$  may depend on the starting point  $\tau_0$ . While multicurves and obstructions are never empty, it is customary to say “ $\Gamma$  is empty” instead of “there is no  $\Gamma$ ” here. Recall the definition of the Thurston matrix  $M_\Gamma$  from Section 2.4. We have:

- If  $\Gamma$  is a simple obstruction,  $M_\Gamma$  has a positive eigenvector  $v$  with eigenvalue  $\lambda_\Gamma \geq 1$ . Suppose that in the Riemann surface  $\pi(\tau)$ , there are annuli around the corresponding geodesics with moduli proportional to  $v$ , then these moduli will grow at least by  $\lambda_\Gamma$  under the pullback  $\pi(\sigma_g(\tau))$ . This is a direct application of the Grötsch inequality [22]. By the collar estimate from Proposition 2.1.3, a lower bound on the modulus corresponds to an upper bound on the hyperbolic length of the geodesic.
- For sufficiently short geodesics, there is a kind of reverse estimate: when  $\Gamma$  is completely invariant but not a simple obstruction, there is a semi-norm on the vector of inverse lengths, which cannot increase arbitrarily. Here a preimage annulus is decomposed along parallels to the core curve, and the new annuli are related to inverse length by the collar theorem again. See Theorem 7.1 in [14], Theorem 10.10.3 in [23], or Lemma 2.6 in [9].

Sketch of the **proof** of Theorem 3.6: Let  $N \geq 0$  be the maximal number of arbitrarily short geodesics in  $\pi(\tau_n)$  as  $n \rightarrow \infty$ . So there is  $R > 0$ , a subsequence  $n_k$ , multicurves  $\Gamma_k$  with  $N$  elements, and  $\varepsilon_k \rightarrow 0$ , such that:

- For each  $n$ , there are at most  $N$  curves  $\gamma$  with  $l(\gamma, \tau_n) < R$ .
- For all  $k$  and all  $\gamma \in \Gamma_k$  we have  $l(\gamma, \tau_{n_k}) < \varepsilon_k$ .

Now if  $N = 0$ , the claims are satisfied for  $\Gamma = \emptyset$ . So assume  $N > 0$ . For large  $k$  we have  $\varepsilon_k \ll R$  and continuity of  $l(\gamma, \cdot)$  together with the reverse inequality above shows that  $\Gamma_k$  is completely invariant. We may assume that all  $\Gamma_k$  have the same partition of marked points  $G \cdot \Gamma_k$ , the same Thurston matrix  $M = M_{\Gamma_k}$ , and  $\pi(\tau_{n_k})$  has a limit in  $\mathcal{S}_{G \cdot \Gamma_k} \subset \widehat{\mathcal{M}}$ . If  $\Gamma_k$  was not a simple obstruction, the reverse inequality applied to  $M$  would give a lower bound for  $\varepsilon_k$ . This is a contradiction for some large  $k$ , and we set  $\Gamma = \Gamma_k$ . Now for all  $n \geq n_k$  the moduli of annuli around  $\gamma \in \Gamma$  have non-decreasing lower bounds and the lengths  $l(\gamma, \tau_n)$  have non-increasing upper bounds; together with the assumptions on the subsequence this gives  $l(\gamma, \tau_n) \rightarrow 0$ . The lower bound  $R$  is satisfied by all other curves, and Mumford compactness according to Proposition 2.2.7 applies to all components of  $\mathcal{S}_{G \cdot \Gamma}$ . If some  $\gamma \in \Gamma$  was intersecting a  $\gamma'$  from another simple obstruction homotopically transversely, there would be an upper bound for  $l(\gamma', \tau_n)$  and a lower bound for  $l(\gamma, \tau_n)$  by the collar theorem. Finally, for a different initial  $\tau_0$  all length  $l(\gamma, \tau_n)$  are changed by a factor bounded above and below, so  $\Gamma$  is independent of  $\tau_0$ . ■

An alternative **proof of the Thurston Theorem 2.7** based on Theorem 3.6: Consider  $\tau_n = \sigma_g^n(\tau_0)$  for some  $\tau_0 \in \mathcal{T}$ . If the canonical obstruction is  $\Gamma \neq \emptyset$ , then  $\pi(\tau_n)$  leaves every compact subset of  $\mathcal{M}$ , so  $\sigma_g$  cannot have a fixed point in  $\mathcal{T}$  and there is no rational map  $f$  equivalent to  $g$ .

Now assume  $\Gamma = \emptyset$ . Then  $\pi(\tau_n)$  stays in a compact subset of  $\mathcal{M}$ . If  $g$  is of type  $(2, 2, 2, 2)$ , it may be obstructed or not, equivalent to a rational map or not. But otherwise some iterate of  $\sigma_g$  is uniformly contracting over the compact set of  $\mathcal{M}$  defined by  $R(\tau_0)$ . So  $\tau_n$  converges to a fixed point, which corresponds to a rational map  $f$ . If  $g$  had a non-canonical simple obstruction  $\tilde{\Gamma}$ , then  $\tau_n$  could not converge to the fixed point if  $\tau_0$  had  $l(\gamma, \tau_0)$  too small for  $\gamma \in \tilde{\Gamma}$ . ■

### 3.3 Characterization of the canonical obstruction

There is no algorithm to determine obstructions of an arbitrary Thurston map  $g$ , but if you have a guess what the canonical obstruction  $\Gamma$  might be, this can be checked with the following criterion. In particular, it shows that the canonical obstruction of a formal mating from non-conjugate limbs is given by loops around ray-equivalence classes with at least two postcritical points; see Section 4.3. Examples of canonical obstructions are given in Figures 2 and 3 as well.

#### Theorem 3.7 (Selinger characterization of the canonical obstruction)

*When  $g$  is a Thurston map of degree  $d \geq 2$ , consider the family of multicurves  $\tilde{\Gamma}$ , which are simple obstructions or empty, with the following property: for the map  $g_{\tilde{\Gamma}}$  between components of the noded surface defined by  $\tilde{\Gamma}$ , the first-return map of each periodic component is*

- a homeomorphism,
- an unobstructed Thurston map, or



- a  $(2, 2, 2, 2)$ -map with a non-pinching obstruction: all curves are essential with respect to the four postcritical points, and the degree of the map is a square. Now this family of multicurves  $\tilde{\Gamma}$  has a unique minimal element with respect to inclusion, which is the canonical obstruction  $\Gamma$ .

Idea of the **proof** from [49]: First, suppose that a first-return map of  $\hat{\mathbb{C}}/\Gamma$  is obstructed, then  $g$  has a non-canonical obstruction  $\Gamma'$ . Suitable annuli of maximal modulus have the property that these moduli are increasing and bounded above. A subsequence of rational maps converges to a limit map on the component in a suitable normalization. This map has annuli of invariant maximal modulus, so the subdivision of preimages happens parallel to the core curves; this fact is used to obtain type  $(2, 2, 2, 2)$ . Now obstructions are related to integer eigenvalues of the corresponding matrix lift, and if the degree was not a square, these eigenvalues would be different and have a quotient  $> 1$ . But then the Thurston matrix of  $\Gamma'$  would have  $\lambda_{\Gamma'} > 1$  and  $\Gamma'$  would be pinching for  $g$  as well.

So  $\Gamma$  satisfies the assumptions on  $\tilde{\Gamma}$  in Theorem 3.7. Conversely, we must see that any simple obstruction  $\tilde{\Gamma}$  with these properties contains the canonical obstruction  $\Gamma$ . Since curves of  $\Gamma$  and  $\tilde{\Gamma}$  do not intersect according to Theorem 3.6, it remains to show that no periodic component of  $\hat{\mathbb{C}} \setminus \tilde{\Gamma}$  contains a curve of  $\Gamma$ . When the first-return map is a Thurston map, this follows from similar arguments as above. When it is a homeomorphism, there would be a Lévy cycle intersecting  $\Gamma$  within the component otherwise. ■

The assumption that  $\tilde{\Gamma}$  is a simple obstruction is necessary, because otherwise curves from  $\tilde{\Gamma}$  and  $\Gamma$  might intersect. Consider the example in Figure 8 of [14], where the spider map of angle  $5/12$  is mated with its conjugate. Various obstructions are formed by the curves  $\alpha, \beta$ , which surround 2-cycles corresponding to fixed points of  $z^2 + \gamma_M(5/12)$ , and by  $\delta_1, \delta_2, \delta_3, \delta_4$ , which surround conjugate postcritical points. Denoting the equator by  $\varepsilon$ ,  $\tilde{\Gamma} = \{\alpha, \beta, \varepsilon\}$  would be a non-simple obstruction with unobstructed component maps, and it does not contain the canonical obstruction  $\Gamma = \{\delta_1, \delta_2\}$ .

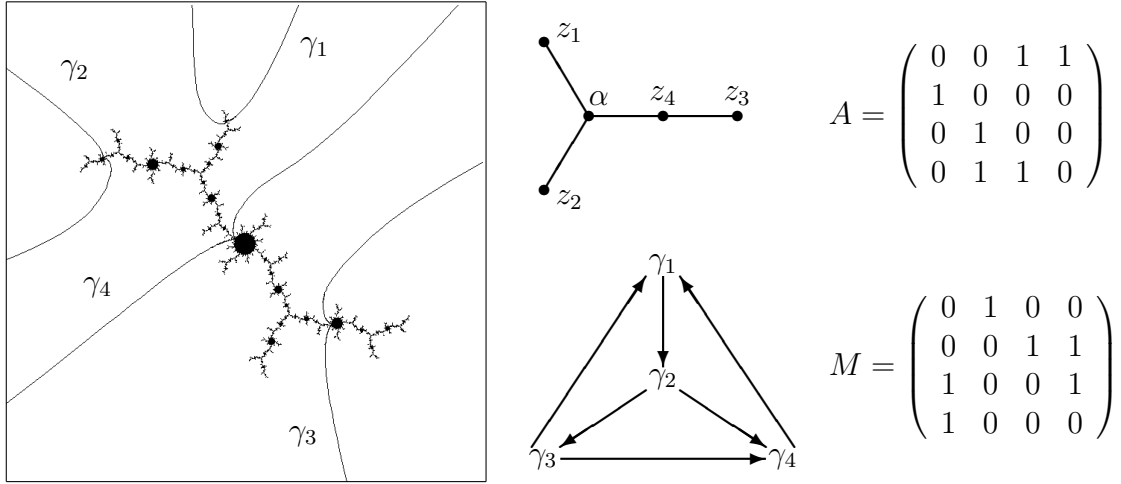
Eigenvalues of non-negative integer matrices appear in two different areas of Thurston's work: the combinatorial characterization of rational maps, and core entropy of quadratic polynomials.  $h(c)$  is the topological entropy of  $z^2 + c$  on its Hubbard tree  $T_c \subset \mathcal{K}_c$ . It is computed from the growth factor of iterated preimages, which in the postcritically finite case is the leading eigenvalue  $1 \leq \lambda \leq 2$  of the Markov matrix  $A$  describing the mapping of edges under the polynomial. Moreover, it is related to the Hausdorff dimension of biaccessing angles. See [25] and the references therein.

**Proposition 3.8 (Mating conjugate polynomials)**

Suppose  $p \neq 0$  is postcritically finite and take the complex conjugate parameter  $q = \bar{p}$ . Consider the canonical obstruction  $\Gamma$  of the formal mating  $g = P \sqcup Q$ . Then:

1. Each  $\gamma \in \Gamma$  passes through a unique edge of the Hubbard tree  $\varphi_0(T_p)$  and through the corresponding edge of  $\varphi_\infty(T_q)$ ; there is a unique  $\gamma \in \Gamma$  for each edge.
2. The Markov matrix  $A$  of  $T_p$  is the transpose of the Thurston matrix  $M = M_\Gamma$  of  $g$ , unless different conventions are used in the preperiodic case, with the critical points marked in the Hubbard tree but not in the formal mating: then  $A$  has an

additional eigenvalue of 0 compared to  $M$ . The leading eigenvalue  $\lambda$  is equal in any case, so  $h(p) = \log \lambda_\Gamma$ .



**Figure 3:** The formal mating of the Kokopelli  $p = \gamma_M(3/15)$  with its conjugate  $q = \bar{p}$ . The 4-periodic rays define four loops  $\gamma_i$  and a noded surface  $\widehat{\mathbb{C}}/\Gamma$  with five pieces. The Thurston matrix  $M$  is the transpose of the Markov matrix  $A$ , which describes the mapping of edges in the Hubbard tree  $T_p$ . The leading eigenvalue  $\lambda = 1.395337$  with  $\lambda^4 - 2\lambda - 1 = 0$  determines the core entropy  $h(p) = \log \lambda$ .

Under pullback with  $g = P \sqcup Q$ , the three Lévy cycles converge to ray-equivalence classes:  $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$  gives the original 4-periodic rays,  $(\gamma_1, \gamma_2, \gamma_3)$  converges to a 3-cycle of loops with 6-periodic rays, and  $(\gamma_1, \gamma_2, \gamma_4)$  gives the 3-periodic rays from  $\alpha_p$  to  $\alpha_q$ .

**Proof** of Proposition 3.8: The Hubbard tree  $T_p$  is a finite tree with expanding dynamics; critical and postcritical points are marked and there may be additional branch points [8]. Let us assume that the critical points are marked as well in the formal mating  $g = P \sqcup Q$ . Marking 1 on the 0-ray will not change the canonical obstruction. Now for each edge of  $T_p$  choose an arc in the dynamic plane of  $P$  passing homotopically transversely through this edge; join it with the complex conjugate arc in the dynamic plane of  $Q$  to obtain a simple loop in  $\widehat{\mathbb{C}}$  separating marked points of  $g$ . This works, since the formal mating is constructed by mapping each dynamic plane to a half-sphere. The curves in Figure 3 are dynamic rays in fact; this choice is possible unless  $p$  is a direct satellite center, but it is not required here.

1. These loops define a non-empty multicurve  $\Gamma$ . Under  $P$ , each edge is covered by either one or two edges, so  $\Gamma$  is invariant: in the former case, one of the two preimages of the corresponding loop is inessential.  $\Gamma$  is completely invariant in fact, since every edge covers at least one edge. By construction, the Markov matrix  $A$  of  $T_p$  and the Thurston matrix  $M_\Gamma$  agree except for transposition. Now  $M_\Gamma$  contains at least one nonzero entry in each row and in each column, so  $\Gamma$  is a simple obstruction.

Looking at noded Riemann surfaces in  $\mathcal{M}_{G,\Gamma}$  or at the noded topological surface defining the pullback map on  $\mathcal{S}_\Gamma$ , the nodes correspond to edges of  $T_p$  and the pieces correspond to the vertices of  $T_p$ , so they contain a single marked point or branch point of  $T_p$  and the corresponding point of  $T_q$ . In the former case, there are two marked points of  $g$  and at least one node, while in the latter case, there is no marked point of  $g$  and at least three nodes.

In the preperiodic case, all first-return maps are homeomorphisms. In the periodic case, the first-return maps for marked points of  $T_p$  map a sphere with three or four marked points to itself. In the latter case the two critical points are fixed and another point is fixed as well, while the fourth one goes to that fixed point. This map is unobstructed by the core arc argument: an obstructing curve cannot separate the two critical points, because its preimage would cover it twice, giving an eigenvalue  $1/2 < 1$ . So an obstruction must surround an arc between the critical points, and its preimages are two curves separating the critical points from one of the other points each, so they are not homotopic to the original curve.

By the Characterization Theorem 3.7,  $\Gamma$  contains the canonical obstruction. It remains to show that no proper subset has the same properties. Assuming that  $M_\Gamma$  and thus  $A$  is reducible in the sense of Perron–Frobenius, it has some block structure understood in terms of simple renormalization according to [25]. For each possible simple renormalization  $p = p' * \hat{p}$ , so  $p$  belongs to a small Mandelbrot set  $p' * \mathcal{M}$  [34], we have a block-triangular structure of  $A$  with two diagonal blocks, one corresponding to the periodic parameter  $p'$ , and an imprimitive one related to the renormalized parameter  $\hat{p}$ . Now  $A$  maps edges from the big Julia set of  $p'$  over edges from the small Julia set of  $\hat{p}$ , and  $M_\Gamma$  maps small loops to big loops. So taking only the small loops would not give an invariant multicurve, and taking only the big loops would give obstructed component maps: the first-return map corresponds to a mating of conjugate polynomials again. Conversely, although not every possible block decomposition is explained in terms of renormalization, every edge of  $T_p$  belongs to a particular level of renormalization, so removing the loop for this edge amounts to removing a block from a particular form with two diagonal blocks.

2. By definition,  $\lambda_\Gamma$  is the leading eigenvalue of  $M_\Gamma$  and  $h(p) = \log \lambda$  with the leading eigenvalue  $\lambda$  of  $A$ . In the preperiodic case, we need not mark the critical points for  $g$ , and we might also not mark it in  $T_p$ ; then we still have  $A$  as the transpose of  $M_\Gamma$ , since the edge around 0 is mapped two-to-one to the edge before  $p$  and the corresponding loop has two homotopic preimages. The matrices will be different using different conventions for  $T_p$  and  $g$ , but according to [25] the leading eigenvalue is the same. ■

Any multicurve corresponds to a tree in a similar way. A general estimate of  $\lambda_\Gamma$  in terms of entropy is given by Shishikura according to [53]. Further applications include the description of Herman rings [51], the construction of maps with Cantor families of circles in the Julia set [18], and the classification of rescaling limits [1]. In recent talks on tropical complex dynamics, Shishikura has suggested to combine arithmetic surgery on a tree with quasiconformal surgery on pieces.

The canonical decomposition of a Thurston map is done in two steps by Bartholdi–Dudko [4]: Lévy cycles generate a decomposition into pieces, such that the first-return maps are homeomorphisms, expanding, or of type  $(2, 2, 2)$ . The expanding pieces are decomposed again, such that the first-return maps are equivalent to rational ones. Here expansion refers to a suitable metric, which is defined everywhere except at super-attracting cycles. See [6, 19] for various notions of expansion.

### 3.4 The Selinger proof of the Pilgrim Conjecture

For a Thurston map  $g$  with canonical obstruction  $\Gamma \neq \emptyset$ , the first-return maps of  $g_\Gamma$  are homeomorphisms or Thurston maps. In [43], Pilgrim has conjectured that the latter are unobstructed when the orbifold is hyperbolic. This was proved by Selinger in Theorem 10.3 of [48] by constructing a subsequence of  $(\tau_n)$  with suitable convergence properties. He obtained the more general Characterization Theorem 5.6 of [49] using different techniques; see Theorem 3.7 in Section 3.3. We will need convergence properties from the first proof in Section 3.5. Maybe the following proof shall be read as a complement to the original one in [48], since it focuses on a few points that took me some time to understand, in particular the construction of the uniform bound  $D_1$  and dealing with the fact that  $\pi_1 : \widehat{\mathcal{T}} \rightarrow \widehat{\mathcal{W}}$  and  $\pi_2 : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{M}}$  are not covering maps.

The Hurwitz space is defined as  $\mathcal{W} = \mathcal{T}/H$ , where  $H < G$  denotes the subgroup of liftable homeomorphisms:  $h \in H$  if there is an  $h' \in G$  with  $g \circ h' = h \circ g$ . Then the covering  $\pi : \mathcal{T} \rightarrow \mathcal{M}$  factorizes as  $\pi = \pi_2 \circ \pi_1$  with  $\pi_1 : \mathcal{T} \rightarrow \mathcal{W}$  and  $\pi_2 : \mathcal{W} \rightarrow \mathcal{M}$ . Moreover,  $H$  has finite index in  $G$  and  $\pi_2$  is a finite cover. Since  $\mathcal{W}$  can be represented by triples of rational maps  $f_\tau$  and marked points in its domain and range [14, 23, 30, 31], there is a continuous map  $\tilde{\sigma}_g : \mathcal{W} \rightarrow \mathcal{M}$  with  $\tilde{\sigma}_g \circ \pi_1 = \pi \circ \sigma_g$ . Now both  $G$  and  $H$  act by WP-isometries on  $\widehat{\mathcal{T}}$ , and we have a completion  $\widehat{\mathcal{W}} = \widehat{\mathcal{T}}/H$  of Hurwitz space, which is compact. While  $\pi, \pi_1, \pi_2$  are covering maps and local isometries, the extensions  $\pi_1 : \widehat{\mathcal{T}} \rightarrow \widehat{\mathcal{W}}$  and  $\pi_2 : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{M}}$  are weak contractions;  $\pi_1$  is “infinitely branched” and  $\pi_2$  is finitely branched. The unique extension  $\tilde{\sigma}_g : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{M}}$  is Lipschitz continuous with factor  $\sqrt{d}$ . The strata of  $\widehat{\mathcal{W}}$  are labeled by classes  $H \cdot \Gamma$  of multicurves. We will not need a concrete description of  $\widehat{\mathcal{W}}$  and  $\tilde{\sigma}_g$  by triples.

#### Proposition 3.9 (Selinger proof of the Pilgrim Conjecture)

Suppose  $g$  is a Thurston map of degree  $d$  with canonical obstruction  $\Gamma \neq \emptyset$ ,  $C$  is a piece of  $\widehat{\mathcal{C}}/\Gamma$  mapped to itself by  $g_\Gamma$ , and the component map  $g^C : C \rightarrow C$  of degree  $\geq 2$  does not have orbifold type  $(2, 2, 2, 2)$ . Fix  $\tau_0 \in \mathcal{T}$  and set  $\tau_n = \sigma_g^n(\tau_0)$ . Then:

1. There is sequence  $w_i$  in a compact subset of  $S_{H \cdot \Gamma} \subset \widehat{\mathcal{W}}$  with  $\tilde{\sigma}_g(w_i) = \pi_2(w_{i+1})$  for  $i \geq 0$ , and a subsequence  $\tau_{n_k}$  with  $\pi_1(\sigma_g^i(\tau_{n_k})) \rightarrow w_i$  as  $k \rightarrow \infty$ , for all  $i \geq 0$ .
2. Assuming that  $\varepsilon(I)$  is sufficiently small and  $\varepsilon(I) \searrow 0$  sufficiently fast, there are  $k(I)$  and  $\hat{\tau}_I \in \pi_1^{-1}(w_0) \cap S_\Gamma$  with  $d_{WP}(\sigma_g^i(\tau_{n_{k(I)}}), \sigma_g^i(\hat{\tau}_I)) < \varepsilon(I)$  for  $0 \leq i \leq I$ , such that  $\pi_1(\sigma_g^i(\hat{\tau}_I)) = w_i$  for  $0 \leq i \leq I$ .
3. The component  $\sigma_C = \sigma_{g^C}$  of the extended pullback map has a unique fixed point  $\hat{\tau}^C$  and  $d_T^C(\sigma_C^i(\hat{\tau}_I^C), \hat{\tau}^C) \rightarrow 0$  as  $i \rightarrow \infty$ , uniformly in  $I \geq i$ .

Intuitively, the situation is as follows: imagine a system of coordinates in a neighborhood of  $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$  adapted to a product of three components. The first coordinate becomes 0 as  $\Gamma$  is pinched. The second coordinate is related to pieces, where the first-return map is a homeomorphism, and where we do not have convergence. The third coordinate describes pieces where the first-return maps converge. Although such a local product representation of  $\widehat{\mathcal{T}}$  is constructed in [41], we do not have any estimates of  $\sigma_g$  in that representation. So the proof will be given by constructing various subsequences in an interplay between  $\mathcal{T}$ ,  $\widehat{\mathcal{T}}$ , and components of  $\mathcal{S}_\Gamma$ , using both

the Teichmüller metric and the Weil–Petersson metric at times. — Note that we have an accumulation statement instead of convergence for two reasons: there may be pieces with a least four marked points and nodes permuted by a homeomorphism, and  $\sigma_g$  is not weakly contracting on  $\widehat{\mathcal{T}}$ .

**Proof:** 1. By the Canonical Obstruction Theorem 3.6,  $\pi(\tau_n)$  accumulates on a compact subset of  $\mathcal{S}_{G,\Gamma} \subset \widehat{\mathcal{M}}$ ; pick an accumulation point  $m_0$ . Now  $\pi_2 : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{M}}$  is a finite branched cover, so there is a subsequence  $(\tau_{n_k^0})_{k \in \mathbb{N}_0} \subset (\tau_n)_{n \in \mathbb{N}_0}$  and a  $w_0 \in \pi_2^{-1}(m_0)$  with  $\pi_1(\tau_{n_k^0}) \rightarrow w_0$  as  $k \rightarrow \infty$ ; we have  $w_0 \in \mathcal{S}_{H,\Gamma} \subset \widehat{\mathcal{W}}$  since precisely the curves in  $\Gamma$  are pinched as  $n_k^0 \rightarrow \infty$ . Then  $\pi(\sigma_g(\tau_{n_k^0})) \rightarrow m_1 = \tilde{\sigma}_g(w_0)$  by continuity. Choose a subsequence  $\tau_{n_k^1}$  of  $\tau_{n_k^0}$  with  $\pi_1(\sigma_g(\tau_{n_k^1})) \rightarrow w_1 \in \pi_2^{-1}(m_1)$ . Define  $m_i, w_i$ , and subsequences  $\tau_{n_k^i}$  inductively, then any subsequence  $\tau_{n_k}$  of the diagonal sequence  $\tau_{n_k^i}$  satisfies the claim.

To obtain the bound  $D_1$  below, this subsequence is constructed as follows: For  $\tau$  on a smooth curve from  $\tau_0$  to  $\tau_1$ , there is a lower bound  $l(\gamma, \sigma_g^n(\tau)) \geq R$ ,  $\gamma \notin \Gamma$ . Take constants  $C_*$  for  $\mathcal{T} = \mathcal{T}_g$  according to item 5 of Proposition 2.2 and  $D_* = D_*(R)$  for  $\mathcal{T}^c = \mathcal{T}_{g^c}$  according to item 6. Pick intermediate points  $\tau_0^0 = \tau_0, \tau_0^1, \dots, \tau_0^U = \tau_1$  on the curve with  $d_{\mathcal{T}}(\tau_0^{u-1}, \tau_0^u) \leq D_*/C_*$  for  $1 \leq u \leq U$ . Now choose the indices  $n_k$  for the subsequence  $\tau_{n_k}$  of  $\tau_{n_k^i}$  such that there are limits  $\pi_1(\sigma_g^{n_k}(\tau_0^u)) \rightarrow w_0^u$  as  $k \rightarrow \infty$ .

2. Intuitively, if  $\mathcal{N}$  is a small neighborhood of  $w_0$  in  $\widehat{\mathcal{W}}$ , we have  $\pi_1(\tau_{n_k}) \in \mathcal{N}$  for sufficiently large  $k$ , and we shall find  $\hat{\tau}_0 \in \pi_1^{-1}(w_0)$  close to  $\tau_{n_k}$  for some  $k$ . Then the curves of  $\tilde{\Gamma} \in G \cdot \Gamma$  are short in a neighborhood of  $\hat{\tau}_0$  and only the curves in  $\Gamma$  are short at  $\tau_{n_k}$ , so  $\tilde{\Gamma} = \Gamma$  and  $\hat{\tau}_0 \in \mathcal{S}_{\Gamma}$ . A subgroup  $G_{\Gamma} < G$  acts transitively on the fiber  $\pi_1^{-1}(w_0) \cap \mathcal{S}_{\Gamma}$ ; it leaves  $\Gamma$  invariant and it is generated by Dehn twists about curves in pieces of  $\widehat{\mathcal{C}}/\Gamma$  and about curves of  $\Gamma$ . The latter act trivially on the stratum but not trivially in a neighborhood, thus  $\pi_1$  is not a covering. See also Remark 3.4. In a neighborhood of  $\hat{\tau}_0$ ,  $\pi_1$  maps to the quotient with respect to  $H \cap G_{\Gamma}$ ,  $\widehat{\mathcal{T}}$  is a product of disks and of half-disks plus the center point, and  $\pi_1$  is infinite-to-one locally on the half-disks at those points. So  $\pi_1$  is not a local WP-isometry at  $\hat{\tau}_0$ : we have  $d_{WP}(\pi_1(\tau'), \pi_1(\tau)) = \min d_{WP}(h \cdot \tau', \tau) \leq d_{WP}(\tau', \tau)$ , and arbitrarily close to  $\hat{\tau}_0$  there are  $\tau', \tau \in \mathcal{T}$  with  $d_{WP}(\pi_1(\tau'), \pi_1(\tau)) < d_{WP}(\tau', \tau)$ . But  $\widehat{\mathcal{T}}$  is a unique geodesic space and the geodesic from  $\tau_{n_k}$  to  $\hat{\tau}_0$  is contained in  $\mathcal{T}$  except for the endpoint [60]. All  $h \cdot \tau_{n_k}$  close to  $\hat{\tau}_0$  are related by global isometries fixing  $\hat{\tau}_0$ , thus

$$d_{WP}(\tau_{n_k}, \hat{\tau}_0) = d_{WP}(\pi_1(\tau_{n_k}), w_0) = d_{WP}(\pi(\tau_{n_k}), m_0). \quad (6)$$

So we may define  $\mathcal{N}$  in terms of a small distance  $\varepsilon(0)$  to  $w_0$  and have the same radius in components of  $\pi_1^{-1}(\mathcal{N})$ . Before stating the actual construction of  $\hat{\tau}_0$ , note that we want to have a shadowing property for a finite number  $I$  of steps; this is possible since  $\sigma_g$  is Lipschitz continuous, and both  $k(I)$  and  $\hat{\tau}_I$  will depend on  $I$ .

Assume that  $\varepsilon(I) \searrow 0$  for  $0 \leq I \rightarrow \infty$  and  $(\sqrt{d} + 1)\varepsilon(I)$  is less than the minimal distance in the fiber  $\pi_2^{-1}(m_i) \cap \mathcal{S}_{H,\Gamma}$  for  $0 \leq i \leq I$ . Moreover, the preimage of an  $\varepsilon(I)$ -neighborhood of  $w_i$  under  $\pi_1$  shall have disjoint components, where  $\pi_1$  is described explicitly as an infinite-to-one map in terms of  $H \cap G_{\Gamma}$  as explained above. Then for  $I \geq 0$  there are  $k(I)$  and  $\hat{\tau}_I \in \pi_1^{-1}(w_0) \cap \mathcal{S}_{\Gamma}$  with  $d_{WP}(\sigma_g^i(\tau_{n_{k(I)}}), \sigma_g^i(\hat{\tau}_I)) < \varepsilon(I)$  for  $0 \leq i \leq I$ . A finite induction shows  $\pi_1(\sigma_g^i(\hat{\tau}_I)) = w_i$  for  $0 \leq i \leq I$ : assuming the

claim for  $i - 1$ , we have  $\pi(\sigma_g^i(\widehat{\tau}_I)) = \widetilde{\sigma}_g(\pi_1(\sigma_g^{i-1}(\widehat{\tau}_I))) = \widetilde{\sigma}_g(w_{i-1}) = m_i$  and

$$\begin{aligned}
& d_{WP}(w_i, \pi_1(\sigma_g^i(\widehat{\tau}_I))) \\
& \leq d_{WP}(w_i, \pi_1(\sigma_g^i(\tau_{n_k}))) + d_{WP}(\pi_1(\sigma_g^i(\tau_{n_k})), \pi_1(\sigma_g^i(\widehat{\tau}_I))) \\
& \leq d_{WP}(m_i, \pi(\sigma_g^i(\tau_{n_k}))) + d_{WP}(\sigma_g^i(\tau_{n_k}), \sigma_g^i(\widehat{\tau}_I)) \\
& \leq d_{WP}(\widetilde{\sigma}_g(\pi_1(\sigma_g^{i-1}(\widehat{\tau}_I))), \widetilde{\sigma}_g(\pi_1(\sigma_g^{i-1}(\tau_{n_k})))) + d_{WP}(\sigma_g^i(\tau_{n_k}), \sigma_g^i(\widehat{\tau}_I)) \\
& \leq (\sqrt{d} + 1)\varepsilon(I).
\end{aligned} \tag{7}$$

Concerning the first term in (7), we have discarded  $\pi_2^{-1}$ , which is not a weak contraction in general. But in this case  $d_{WP}(w_i, \pi_1(\sigma_g^i(\tau_{n_k}))) = d_{WP}(m_i, \pi(\sigma_g^i(\tau_{n_k})))$  by the same arguments as for (6). Finally, by the assumption on the distance in the fiber  $\pi_2^{-1}(m_i)$ , the claim is proved for  $i$ .

3. All maps, sets, and elements related to the stratum  $\mathcal{S}_\Gamma \in \widehat{\mathcal{T}}$  have a product structure describing pieces of the noded Riemann surfaces; the component for the piece corresponding to  $C$  is denoted by a superscript or subscript  $C$ . Since the length of hyperbolic geodesics is continuous on  $\widehat{\mathcal{T}}$ , we have  $l(\gamma, \sigma_c^i(\widehat{\tau}_I^C)) \geq R$  for  $0 \leq i \leq I$  and all essential simple closed curves  $\gamma$  in  $C$ . The same lower bound holds for simple closed geodesics in the corresponding piece of the noded surface defined by  $\pi_2(w_0^u)$ . Now  $\sigma_g$  is weakly contracting on  $\mathcal{T}$  with respect to  $d_T$ , so the intermediate points satisfy  $d_T(\sigma_g^{n_k}(\tau_0^{u-1}), \sigma_g^{n_k}(\tau_0^u)) \leq D_*/C_*$  and  $d_{WP}(\sigma_g^{n_k}(\tau_0^{u-1}), \sigma_g^{n_k}(\tau_0^u)) \leq D_*$ . For  $I \geq I_*$  there are  $\widehat{\tau}_I^u \in \pi_1^{-1}(w_0^u) \cap S_\Gamma$  with  $d_{WP}(\sigma_g^{n_k(I)}(\tau_0^u), \widehat{\tau}_I^u) < \varepsilon(I) \leq D_*/2$ . Then  $d_{WP}(\widehat{\tau}_I^{u-1}, \widehat{\tau}_I^u) < 2D_*$  and the same estimate holds for the components related to  $C$ . Since these components are intermediate points between  $\widehat{\tau}_I^C$  and  $\sigma_c(\widehat{\tau}_I^C)$  with geodesic length  $\geq R$ , item 6 of Proposition 2.2 gives

$$d_T(\widehat{\tau}_I^C, \sigma_c(\widehat{\tau}_I^C)) < 2U \cdot 1/4 = U/2 \leq D_1. \tag{8}$$

Here  $D_1 \geq U/2$  is chosen such that the estimate holds not only for  $I \geq I_*$  but for all  $I \geq 1$ . Now suppose  $0 \leq i < I$ . The pullback map  $\sigma_c$  is weakly contracting, so  $\sigma_c^i(\widehat{\tau}_I^C)$  and  $\sigma_c^{i+1}(\widehat{\tau}_I^C)$  are connected with an arc of T-length  $\leq D_1$ , on which the length of simple closed geodesics is bounded below by  $R \cdot e^{-D_1}$ . This condition defines compact subsets of  $\mathcal{M}_C$  and  $\mathcal{W}_C$  according to the Mumford Proposition 2.2.7. The contraction of  $\sigma_c$  at  $\tau^C$  depends only on  $f_{\tau^C}$  or  $\pi_1^C(\tau^C)$ , so some iterate of  $\sigma_c$  is uniformly strictly contracting over the compact subset of  $\mathcal{W}_C$ , since  $g^C$  is not of type  $(2, 2, 2, 2)$ . For notational convenience, let us assume that the first iterate suffices. So there is  $0 < L < 1$  with

$$d_T(\sigma_c^i(\widehat{\tau}_I^C), \sigma_c^{i+1}(\widehat{\tau}_I^C)) \leq D_1 \cdot L^i \quad \text{and} \quad d_T(w_i^C, w_{i+1}^C) \leq D_1 \cdot L^i \tag{9}$$

for  $0 \leq i < I$ ; the second estimate is true for all  $i \geq 0$  since  $w_i$  is independent of  $I > i$ . By completeness of  $\mathcal{W}^C$  we have limits  $w_i^C \rightarrow \widehat{w}^C$  and  $m_i^C \rightarrow \widehat{m}^C$  with  $\widetilde{\sigma}_c(\widehat{w}^C) = \pi_2^C(\widehat{w}^C) = \widehat{m}^C$  and an estimate

$$d_T(w_i^C, \widehat{w}^C) \leq \frac{D_1}{1-L} L^i. \tag{10}$$

Take a lower bound  $\widehat{\varepsilon}$  for the T-distance in the fiber  $(\pi_1^C)^{-1}(\widehat{w}^C)$ , and such that an  $\widehat{\varepsilon}/3$ -neighborhood of  $\widehat{w}^C$  has disjoint preimages under the covering  $\pi_1^C$ . Choose

$i = i_*$  such that the right hand side of (10) is less than  $\hat{\varepsilon}/3$ . Pick  $I > i$  and find  $\hat{\tau}^C \in (\pi_1^C)^{-1}(\hat{w}^C)$  with  $d_T(\sigma_c^i(\hat{\tau}_I^C), \hat{\tau}^C) < \varepsilon/3$ . Then

$$d_T(\hat{\tau}^C, \sigma_c(\hat{\tau}^C)) \leq d_T(\hat{\tau}^C, \sigma_c^i(\hat{\tau}_I^C)) + d_T(\sigma_c^i(\hat{\tau}_I^C), \sigma_c^{i+1}(\hat{\tau}_I^C)) + d_T(\sigma_c^{i+1}(\hat{\tau}_I^C), \sigma_c(\hat{\tau}^C)), \quad (11)$$

which is bounded by  $\hat{\varepsilon}/3 + \hat{\varepsilon}/3 + \hat{\varepsilon}/3$ . So  $\hat{\tau}^C$  is a fixed point of  $\sigma_c$ . Since the orbifold is not of type  $(2, 2, 2, 2)$ ,  $g^C$  is unobstructed, the fixed point is unique, and with  $D = D_1/(1 - L)$  we have  $d_T(\sigma_c^i(\hat{\tau}_I^C), \hat{\tau}^C) \leq D \cdot L^i$  for all  $i_* \leq i < I < \infty$ . Actually, an estimate of this form is valid as well, when only some iterate of  $\sigma_c$  is strictly contracting. ■

### 3.5 Essential equivalence and convergence properties

Suppose  $g$  is a Thurston map with canonical obstruction  $\Gamma \neq \emptyset$ . Considering the Thurston Algorithm  $\tau_n = \sigma_g^n(\tau_0)$  with rational maps  $f_n$  sending point configurations at time  $n + 1$  to those at time  $n$ , we know that curves corresponding to  $\Gamma$  will be pinched and points will collide. Normalizing three points to  $\infty, 0, 1$ , a component  $C$  of  $\hat{\mathbb{C}} \setminus \Gamma$  is singled out, and all other components will shrink to points. We shall see that in a fairly general situation, these points do not wander, but they have limits as  $n \rightarrow \infty$ . So there is a limiting point configuration, such that not all points are distinct, and a rational map  $f$  with  $f_n \rightarrow f$ . The following Theorem 3.11 has applications to quadratic matings, anti-matings, precaptures, and spiders. Its proof is based on the extension of  $\sigma_g$  to augmented Teichmüller space  $\hat{\mathcal{T}}$ ; but I have tried to formulate the assumptions in terms of components of  $\hat{\mathbb{C}} \setminus \Gamma$  instead of pieces of  $\hat{\mathbb{C}}/\Gamma$ , so that it may be applied in other papers without introducing  $\hat{\mathcal{T}}$ .

#### Definition 3.10 (Essential equivalence)

Consider a bicritical Thurston map  $g : S^2 \rightarrow S^2$  or  $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  of degree  $d \geq 2$ , with marked set  $Z$  including the critical points. Suppose there is a completely invariant multicurve  $\Gamma \neq \emptyset$  and a component  $C$  of  $\hat{\mathbb{C}} \setminus \Gamma$  such that:

- All components  $\tilde{C} \neq C$  of  $\hat{\mathbb{C}} \setminus \Gamma$  are disks; these disks are preperiodic or periodic under  $g$  after an isotopic deformation  $\varphi$ , and the periodic disks are mapped homeomorphically.
- $C$  is mapped to itself with degree  $d$  in the sense explained below; the essential map  $\tilde{g}$  is equivalent to a rational map  $f$ .

Then we shall say that  $g$  is **essentially equivalent** to  $f$ .

Denoting a union of concrete curves by the same symbol  $\Gamma$ , the full preimage is  $g^{-1}(\Gamma) = \Gamma' \cup \Gamma''$ , where  $\Gamma'$  consists of essential curves ambient isotopic to  $\Gamma$  and the curves in  $\Gamma''$  are inessential. Choose a homeomorphism  $\varphi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  with  $\varphi : \Gamma \rightarrow \Gamma'$ , isotopic to the identity relative to the marked set  $Z$ , then  $g \circ \varphi : \Gamma \rightarrow \Gamma$ . We have assumed that  $\Gamma$  is not nested: there is a distinguished component  $C$  of  $\hat{\mathbb{C}} \setminus \Gamma$ , such that all of the other components are disks, and mapped to disks by  $g \circ \varphi$ . Intuitively, the construction of noded topological surfaces is done by pinching all curves in  $\Gamma$ , respectively  $\Gamma' \cup \Gamma''$ , to points and by chopping off spheres with at most one marked point; suitable identifications are needed to obtain a self-map  $g_\Gamma$ .

In this situation, the definition of the component map  $g^C$  according to [48] and Section 3.2 can be paraphrased as follows:  $C' = (g \circ \varphi)^{-1}(C) \subset C$  is a component of

$\widehat{\mathbb{C}} \setminus (\Gamma \cup \varphi^{-1}(\Gamma''))$ . Considering each boundary curve as a single point,  $g \circ \varphi : C' \rightarrow C$  defines branched coverings  $S^2 \rightarrow S^2$  or  $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  up to isotopy. More precisely, choose a continuous  $\psi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ , which is sending closed disk components to points and is injective otherwise. Modify  $\psi$  to  $\psi'$  in disjoint neighborhoods of the additional disk components of  $\widehat{\mathbb{C}} \setminus \varphi^{-1}(\Gamma'')$ , sending these to points as well, without moving possible single marked points in these disks. Then the essential map  $\tilde{g}$  with  $\psi \circ g \circ \varphi = \tilde{g} \circ \psi'$  is a Thurston map, whose marked points correspond to marked points of  $g$  in  $C'$ , disk components of  $\widehat{\mathbb{C}} \setminus \Gamma$ , or to disks bounded by peripheral curves in  $\varphi^{-1}(\Gamma'')$ . It is defined uniquely up to combinatorial equivalence and represents  $g^C$ .

**Theorem 3.11 (Convergence of marked points and rational maps)**

*Consider a bicritical Thurston map  $g$ , a multicurve  $\Gamma \neq \emptyset$  and a component  $C$  of  $\widehat{\mathbb{C}} \setminus \Gamma$ , such that  $g$  is essentially equivalent to a rational map  $f$  according to Definition 3.10. Use a normalization of critical points at 0 and  $\infty$ , and another marked point at 1, which is arbitrary in  $C$  or in a disk not containing a critical point. Normalize  $f$  analogously. Then the Thurston Algorithm  $\sigma_g$  for the unmodified map  $g$  with any initial  $\tau_0 \in \mathcal{T}$  satisfies:*

1. *Precisely the curves of  $\Gamma$  are pinched, and  $\tau_n = \sigma_g^n(\tau_0)$  diverges in  $\mathcal{T}$ .*
2. *If  $f$  is not of type  $(2, 2, 2, 2)$ , we have  $f_n \rightarrow f$ . The marked points converge to preperiodic and periodic points of  $f$ ; two points collide if and only if they belong to the same disk  $\tilde{C} \neq C$  in  $\widehat{\mathbb{C}} \setminus \Gamma$ .*

*Analogous statements hold for a path  $\tau_t$  with  $\tau_{t+1} = \sigma_g(\tau_t)$ .*

The notion of essential equivalence is meant to indicate a generalization of Thurston's combinatorial equivalence, but in this form it is not an equivalence relation. This notion shall emphasize that  $g$  itself determines the canonical obstruction  $\Gamma$ , the essential map  $\tilde{g}$ , and the rational map  $f$ , and no modification is needed to ensure  $f_n \rightarrow f$ .

A typical example is provided by a formal mating  $g = P \sqcup Q$  of quadratic polynomials, having ray connections between postcritical points but no cyclic ray connections: then  $\Gamma$  consists of curves around postcritical ray-equivalence classes. These are the only obstructions according to [54], so the essential mating  $\tilde{g}$  is unobstructed and combinatorially equivalent to a rational map  $f$ , excluding type  $(2, 2, 2, 2)$  for now. Then Theorem 3.11 gives  $f_n \rightarrow f$  for the pullback defined by the unmodified formal mating; see Section 4.3 for details. According to [27], the convergence statement is wrong in general when  $\tilde{g}$  and  $f$  are of type  $(2, 2, 2, 2)$ . — Further applications are given in [11, 28, 29]. For simplicity I have considered the bicritical case only, because  $f_n$  is determined explicitly by its critical values; to obtain a more general result, it should also be checked whether collisions between critical points are allowed.

**Proof of Theorem 3.11.1:** According to the discussion of Theorem 3.5,  $C$  is considered as a piece of a noded surface and is related to a component of the invariant stratum  $\mathcal{S}_{G,\Gamma} \subset \widehat{\mathcal{M}}$ , and  $\tilde{g} = g^C$  is a component map of  $g_\Gamma$ . All disks correspond to pieces, which are attached directly to  $C$ . If one of these pieces contains a critical point, the node will be critical as well, and the piece is mapped with degree  $d$ ; by assumption it is preperiodic. Accordingly, a periodic critical point must belong to  $C$ . Both critical points are allowed to be in pieces corresponding to disks  $\tilde{C} \neq C$ , but not in the same piece, because then  $\tilde{g}$  would have degree one.



By assumption,  $\Gamma$  consists of one or several Lévy cycles and all of their essential preimages, so it is a simple obstruction. The component map  $g^C = \tilde{g}$  is unobstructed, since it is combinatorially equivalent to the rational map  $f$ , which is not a flexible Lattès map. The remaining first-return maps are homeomorphisms, so according to the Characterization Theorem 3.7,  $\Gamma$  contains the canonical obstruction. Assuming that the canonical obstruction was smaller, we would enlarge  $C$  by omitting one or several Lévy cycles from  $\Gamma$ , and  $g^C$  would be obstructed. So  $\Gamma$  is the canonical obstruction of  $g$ ; by Theorem 3.6,  $(\tau_n)$  accumulates at most on  $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$  and  $\pi(\tau_n)$  accumulates on a compact subset of  $\mathcal{S}_{G,\Gamma}$ . ■

Intuitively, what happens is that point configurations  $x_i(n)$  cluster according to the disks containing  $z_i$ , and the pullback of clusters determined by  $\sigma_g$  should stay close to the pullback of single points obtained from  $\sigma_f$ . This argument involves interchanging limits, and I have not been able to prove it with direct estimates. So, convergence will be proved below in two steps: first use augmented Teichmüller space and the Selinger Proposition 3.9 for a global result, accumulation at the prospective limit  $x^\infty$ . Then a local result is applied, attraction according to Proposition 3.12. The global result is needed to get close to  $x^\infty$  and to distinguish different fixed points of the extended pullback relation. And the local result is needed because the global one provides accumulation only, not convergence, as explained in Section 3.4. The pullback of point configurations may be visualized as a kind of movie: as time  $t$  or  $n$  flows, the points  $x_i(n)$  move within a single copy of  $\widehat{\mathbb{C}}$ . To formulate neighborhoods, convergence, or derivatives of point configurations, it is more convenient to consider  $x = (x_1, \dots, x_{|Z|})$  as a single point in  $\widehat{\mathbb{C}}^{|Z|}$ .

**Notations and basic properties** of the pullback relation: The branch portrait of  $g$  defines a map  $\#$  of indices, such that marked points of  $g$  are mapped as  $g(z_i) = z_{i\#}$ . The Thurston Algorithm provides sequences of homeomorphisms  $\psi_n$  and rational maps  $f_n$  with  $f_n \circ \psi_{n+1} = \psi_n \circ g$ . The point configuration  $x(n)$  at time  $n$  has the components  $x_i(n) = \psi_n(z_i)$ ; sometimes these are called marked points as well. The basic relation is  $f_n(x_i(n+1)) = x_{i\#}(n)$ .

The coordinates depend on the normalization, and we have specific indices  $\alpha', \beta', \gamma'$  with  $x_{\alpha'}(n) = \infty$ ,  $x_{\beta'}(n) = 0$ , and  $x_{\gamma'}(n) = 1$  for all  $n$ . Assume that  $z_{\alpha'}$  and  $z_{\beta'}$  are the critical points of  $g$ . The normalization fixes an embedding  $\pi_3 : \mathcal{M} \rightarrow \widehat{\mathbb{C}}^{|Z|}$ ; its range consists of  $(x_1, \dots, x_{|Z|})$  with pairwise distinct components, and  $\infty, 0, 1$  at specific positions, which is open and dense in a subset  $\mathbb{C}^{|Z|-3} \subset \widehat{\mathbb{C}}^{|Z|}$ .

Denote  $\alpha = \alpha'\#, \beta = \beta'\#, \gamma = \gamma'\#$ , then  $f_n$  is determined by  $f_n(\infty) = x_\alpha(n)$ ,  $f_n(0) = x_\beta(n)$ , and  $f_n(1) = x_\gamma(n)$  as a Möbius transformation of  $z^d$ . This gives the pullback relation in the form of a multi-valued function from  $\pi_3(\mathcal{M})$  into itself:

$$f_n^{-1}(z) = \sqrt[d]{\frac{x_\gamma(n) - x_\alpha(n)}{x_\gamma(n) - x_\beta(n)} \cdot \frac{z - x_\beta(n)}{z - x_\alpha(n)}} \quad x'_i = \sqrt[d]{\frac{x_\gamma - x_\alpha}{x_\gamma - x_\beta} \cdot \frac{x_{i\#} - x_\beta}{x_{i\#} - x_\alpha}} \quad (12)$$

The second formula may be considered either as a multi-valued function  $x \mapsto x'$ , or as a step of the pullback  $x(n) \mapsto x(n+1)$ . By the normalization we have  $|Z| - 3$  independent variables in the domain and  $|Z| - 3$  variables in the range; when some  $x_m = \infty$ , the corresponding factors cancel from the radicand. For each value of the index  $i$ , the radicand is either constant  $\infty, 0, 1$  or never  $\infty, 0, 1$ . In the Thurston pullback, a specific branch of the  $d$ -th root is chosen by the isotopy class of  $\psi_{n+1}$

or by continuity of the path  $x_i(t)$ . Note that unless all marked points  $z_i$  of  $g$  are periodic, we have pairs of indices  $i \neq k$  with  $g(z_i) = g(z_k)$ , so  $i\# = k\#$ . Then  $x_i(n+1)$  and  $x_k(n+1)$  are given by different branches of a root with the same radicand, and the number of variables could be reduced by replacing  $x_k$  with  $\zeta \cdot x_i$  for some  $\zeta$  with  $\zeta^d = 1$  determined by  $g$ ; while this makes sense for a concrete example, it would complicate the notation for the present discussion.

**Relating  $g$  to  $f$ :** To describe the Thurston pullback  $\sigma_f$ , denote the marked points of  $f$  by  $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{|\tilde{Z}|})$  and define indices with  $\tilde{z}_{\alpha'} = \infty$ ,  $\tilde{z}_{\beta'} = 0$ ,  $\tilde{z}_{\gamma'} = 1$ , and  $\tilde{z}_{\alpha} = f(\infty)$ ,  $\tilde{z}_{\beta} = f(0)$ ,  $\tilde{z}_{\gamma} = f(1)$ . Each marked point of  $f$  corresponds either to a unique marked point of  $g$  in  $C$  or to a unique disk component  $\tilde{C} \neq C$  of  $\hat{\mathbb{C}} \setminus \Gamma$  with at least two marked points; this defines a surjection  $D : \{1, \dots, |Z|\} \rightarrow \{1, \dots, |\tilde{Z}|\}$ . The normalizations are assumed to be compatible, i.e.,  $D(\alpha') = \tilde{\alpha}'$ ,  $D(\beta') = \tilde{\beta}'$ , and  $D(\gamma') = \tilde{\gamma}'$ .

Denote a specific diagonal of  $\hat{\mathbb{C}}^{|Z|}$  by  $\Delta_\Gamma$ : it contains all  $x = (x_1, \dots, x_{|Z|})$  with  $x_i = x_k$  if and only if  $D(i) = D(k)$ , and  $x_{\alpha'} = \infty$ ,  $x_{\beta'} = 0$ ,  $x_{\gamma'} = 1$ . The extension of  $\pi_3$  to  $\widehat{\mathcal{M}}$  satisfies  $\pi_3(\mathcal{S}_{G,\Gamma}) = \Delta_\Gamma$ ; this is an isomorphism when all disks have at most two marked points, but it is forgetful otherwise. Note that there is a natural bijection between  $\Delta_\Gamma \subset \hat{\mathbb{C}}^{|Z|}$  and the configuration space of  $\sigma_f$  contained in  $\hat{\mathbb{C}}^{|\tilde{Z}|}$ : the components of  $x$  have repetitions, such that  $x$  corresponds to  $\tilde{x}$  with  $x_i = \tilde{x}_{D(i)}$  for all  $i$ . In particular, the point configuration  $x^\infty$  with repetitions given by  $x_i^\infty = \tilde{z}_{D(i)}$  is the projective limit of  $x(n)$ .

The Thurston pullback  $\sigma_f$  defines a multi-valued pullback relation on its configuration space, which is given by formulas analogous to (12). A specific branch of this pullback relation has a fixed point at  $\tilde{x} = \tilde{z}$ ; it is analytic in a neighborhood of  $\tilde{z}$  in  $\mathbb{C}^{|\tilde{Z}|-3} \subset \hat{\mathbb{C}}^{|\tilde{Z}|}$ . A simple but suggestive observation is the following: under the bijection of  $x$  and  $\tilde{x}$  the conjugate pullback map in a neighborhood of  $x^\infty$  in  $\Delta_\Gamma$  is given by choosing a suitable branch in (12). The reason is that when  $z_i$  is in the disk corresponding to  $\tilde{z}_j$ , then  $g(z_i)$  is in the disk corresponding to  $f(\tilde{z}_j)$ . And if  $z_i$  and  $z_k$  are in the same disk, their images are both in one disk. So for  $x \in \Delta_\Gamma$  the radicands with different indices  $i\#$  and  $k\#$  agree whenever  $D(i) = D(k)$ . Note that this does not mean that the local branch extends to a neighborhood of  $x^\infty$  in  $\hat{\mathbb{C}}^{|Z|-3}$ : this is the case precisely when the radicand cannot become 0 or  $\infty$ . If, e.g., there is an index  $k \neq \beta'$  with  $D(k) = D(\beta') = \tilde{\beta}'$ , so  $x_k$  will be identified with 0, the radicand may be 0 within any neighborhood of  $x^\infty$  in  $\hat{\mathbb{C}}^{|Z|-3}$ . (On  $\Delta_\Gamma$ , the radicand in (12) is constant 0 both for  $i = k$  and  $i = \beta'$ , which is not a problem.)

### Proposition 3.12 (Local attraction)

*Consider  $g$ ,  $\Gamma$ , and  $f$  according to Theorem 3.11. Assume in addition that the marked point normalized to 1 is chosen more restrictively: if a marked point is identified with a critical point, then 1 represents a critical value, whose iterates are not identified with a critical point. Using the notations introduced above we have:*

1. *If no marked point is identified with a critical point, a branch of the pullback relation (12) extends analytically to a neighborhood of  $x^\infty$  in  $\mathbb{C}^{|Z|-3}$ . The eigenvalues  $\lambda$  of the derivative at the fixed point  $x^\infty$  are of the following form: they are eigenvalues of  $D\sigma_f$  at [id], or  $\lambda = 0$ , or  $\lambda^{rp} = \rho^{-r}$  when an  $rp$ -cycle of  $g$  in a  $p$ -cycle of disks corresponds to a  $p$ -cycle of  $f$  having multiplier  $\rho$ .*

2. If  $f$  is not of type  $(2, 2, 2, 2)$ , there is a neighborhood  $\mathcal{N}$  of  $x^\infty$  in  $\widehat{\mathbb{C}}^{|Z|-3}$ , which is attracting in the following sense: when  $\tau_t$  is a path in  $\mathcal{T}$  with  $\tau_{t+1} = \sigma_g(\tau_t)$ , and  $\pi_3(\pi(\tau_t)) \in \mathcal{N}$  for a segment of  $t$ -length 1, then the path in configuration space will stay in  $\mathcal{N} \cap \pi_3(\mathcal{M})$  forever and converge to  $x^\infty \in \mathcal{N} \setminus \pi_3(\mathcal{M})$ .

The additional assumption restricts the choice of the index  $\gamma'$  with  $x_{\gamma'}(n) = \psi_n(z_{\gamma'}) = 1$ , when a critical point  $\omega$  of  $g$  is identified with another marked point, i.e., they are in the same disk component  $\tilde{C} \neq C$  of  $\widehat{\mathbb{C}} \setminus \Gamma$ . Then  $\omega$  is strictly preperiodic and all forward iterates of  $\omega$  will belong to disks as well; preimages of  $\omega$  may be marked or not, and in the former case, may undergo identifications or not. In general we may take the critical value  $g(\omega)$  for  $z_{\gamma'}$ , unless it equals the other critical point or is identified with it or with a preimage. In that case we can take the other critical value, which is not identified with a preimage of  $\omega$ , since the critical points of  $f$  are not periodic.

**Proof of Proposition 3.12:** Now after choosing  $\gamma'$  and  $\tilde{\gamma}' = D(\gamma')$ , which determines the term of  $f$ , we shall consider three cases of increasing complexity:

Case 1: no marked point is identified with a critical point.

Case 2: a marked point is identified with a critical point, but no postcritical point is identified with a critical point.

Case 3: for some  $k \geq 1$ ,  $g^k(z_{\alpha'})$  is identified with  $z_{\beta'}$ . This means that  $f^k(\infty) = 0$ , but  $g$  has disjoint critical orbits.

When  $g^k(z_{\beta'})$  is identified with  $z_{\alpha'}$  instead, this does not require separate arguments, since  $f$  and  $f_n$  are related to case 3 by a conjugation with  $z \mapsto 1/z$ .

**Case 1:** Recall that  $\tilde{z}$  denotes the marked points  $\tilde{z}_j$  of  $f$  and  $x^\infty \in \Delta_\Gamma$  with  $x_i^\infty = \tilde{z}_{D(i)}$  is our prospective limit of  $x(n)$ . Since the marked points of  $f$  are preimages of marked points under pullback with suitable branches of  $f^{-1}$ , for each  $i \neq \alpha', \beta'$  there is a unique branch in (12), such that taking components of  $x = x^\infty$  for the radicand, gives  $x'_i = x_i^\infty$  for the root. These branches extend analytically to  $x$  in a neighborhood of  $x^\infty$  in  $\mathbb{C}^{|Z|-3}$ , which means  $|x_i - x_i^\infty| < \varepsilon$  for  $i = 1, \dots, |Z|$  with  $i \neq \alpha', \beta', \gamma'$ , since the radicands are varying in small neighborhoods of values distinct from 0 and  $\infty$ .

To determine the eigenvalues of the derivative matrix for this branch of the pullback relation, we shall obtain block matrices by choosing new coordinates labeled  $u = (u_1, \dots, u_{|\tilde{Z}|})$  and  $v = (v_{|\tilde{Z}|+1}, \dots, v_{|Z|})$  as follows:

- For each  $j = 1, \dots, |\tilde{Z}|$ , choose one index  $i$  with  $D(i) = j$  and set  $x_i = u_j$ . If there are  $k \neq i$  with  $D(k) = j$ , set  $x_k = u_j + v_m$  for an unused index  $m$ . When all variables are defined successively, note that  $x = x^\infty$  corresponds to  $u = \tilde{z}$  and  $v = 0$ .
- For  $j = \tilde{\gamma}'$ , the marked point at 1 shall be  $u_j$ , i.e.,  $x_{\gamma'} = u_{\tilde{\gamma}'}$ . The corresponding choice for 0 and  $\infty$  is satisfied here anyway, because these are not identified with other marked points, but it is required explicitly in cases 2 and 3.
- Renumber  $v_{|\tilde{Z}|+1}, \dots, v_{|Z|}$  such that preperiodic marked points  $u_j + v_m$  appear before periodic ones, higher preperiods before lower preperiods, and the periodic marked points are grouped according to their cycles, with a natural order within each cycle. We may renumber the components of  $x$  such that  $x_j = u_j$  or  $x_i = u_{D(i)} + v_i$ , respectively.

Since three components of  $u$  are constant, the local branch  $(u, v) \mapsto (u', v')$  of the pullback relation still has  $(|\tilde{Z}| - 3) + (|Z| - |\tilde{Z}|) = |Z| - 3$  free variables. We shall see that the derivative at the fixed point  $(\tilde{z}, 0)$  has the block-triangular form

$$D = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \text{with} \quad B = \begin{pmatrix} R & Q \\ 0 & P \end{pmatrix}. \quad (13)$$

Note that the local manifold with  $u \approx \tilde{z}$  and  $v = 0$  is invariant under the pullback relation:  $u'_j$  and  $u'_j + v'_i$  correspond to marked points of  $g$  in the same disk, so their images belong to one disk as well. If  $v = 0$ , the radicands determining  $u'_j$  and  $u'_j + v'_i$  agree, and the local branches of the roots agree, so  $v'_i = 0$ . This argument shows that the lower left block of  $D$  is 0, and the upper left block  $A$  coincides with the derivative of the local pullback relation for  $\sigma_f$  in configuration space, which is analytically conjugate to  $\sigma_f$  in a neighborhood of its fixed point in Teichmüller space. So  $A$  has attracting eigenvalues  $\lambda$  unless  $f$  is of type  $(2, 2, 2, 2)$  — then one eigenvalue will be neutral.

Before discussing the block structure of  $B$ , let us consider partial derivatives of the pullback relation (12) more explicitly. A few components of  $(u, v)$  determine the bicritical rational map  $f_{uv}$  by its critical values and the image of 1. For suitable combinations of indices we have

$$f_{uv}(u'_j + v'_i) = u_l + v_k \quad \text{and} \quad f_{uv}(u'_j) = u_l + v_m \quad (14)$$

in general; either  $v_k$  or  $v_m$  may be missing. Total differentials at  $(u, v) = (\tilde{z}, 0)$  read

$$\dots + f'(\tilde{z}_j) \cdot (du'_j + dv'_i) = du_l + dv_k \quad \text{and} \quad \dots + f'(\tilde{z}_j) \cdot du'_j = du_l + dv_m, \quad (15)$$

where  $\dots$  denotes differentials involving the partial derivatives of  $f_{uv}(z)$  with respect to the parameters  $(u, v)$ . Observing that these expressions agree for both equations when setting  $u = u' = \tilde{z}$  and  $v = v' = 0$ , the difference gives

$$f'(\tilde{z}_j) \cdot dv'_i = dv_k - dv_m. \quad (16)$$

Note again that either  $v_k$  or  $-v_m$  may be missing. The argument remains valid, if a component  $v_i, v_k, v_m$  appears in the parameters of  $f_{uv}$  as well, or if  $u'_j$  or  $u_l$  is 1. Now (16) shows again that the lower left block of  $D$  is 0. The blocks  $R$  and  $P$  of  $B$  in (13) refer to preperiodic and periodic marked points, respectively. In (16),  $v'_i$  has a higher preperiod than  $v_k$  and  $v_m$ , or these refer to a cycle of disks. Thus  $R$  is strictly upper triangular, with 0 on the diagonal, and its eigenvalues  $\lambda$  vanish.

Finally, consider the blocks of  $P$ , which are related to periodic cycles of  $g$ . Since periodic disks of  $\hat{\mathbb{C}} \setminus \Gamma$  are mapped homeomorphically by  $g \circ \varphi$ , each disk in a cycle has the same number of marked points, which are permuted by the first-return map. The obvious examples are two cycles of periodic points with the same period identified pairwise, or a single cycle of high period identified such that a cycle of lower period results for  $f$ . All possibilities are covered by the following description: a cycle of  $f$  has period  $p \geq 1$  and there are one or several cycles of  $g$  with periods  $rp$  for possibly different values  $r \geq 1$ . Consider two scenarios:

First,  $g$  has a  $p$ -cycle in the disks under consideration, which is labeled  $u_1, \dots, u_p$ . The indices are chosen to illustrate the order in the  $v$ -blocks; they do not actually

start with 1. Every other  $rp$ -cycle of  $g$ ,  $r \geq 1$ , is described by  $v_i$  in the following order:  $u_1 + v_1 \dots u_p + v_p, u_1 + v_{p+1} \dots u_p + v_{2p}, \dots, u_1 + v_{(r-1)p+1} \dots u_p + v_{rp}$ . Then  $P$  contains an  $rp$ -block on the diagonal with nonzero entries  $1/f'(\tilde{z}_j)$  only directly above the diagonal and in the lower left corner, since  $\pm v_m$  is absent from (14)–(16). By rescaling all variables, this block becomes a companion matrix and the entry in the lower left position is  $(1/f'(\tilde{z}_1) \dots 1/f'(\tilde{z}_p))^r = \rho^{-r}$ . So the characteristic polynomial is  $\lambda^{rp} - \rho^{-r}$ . Note that  $|\rho| > 1$  because  $f$  is postcritically finite and all periodic points are superattracting or repelling, so  $|\lambda| < 1$ .

Second, all cycles of  $g$  within the  $p$ -cycle of disks have periods  $rp$  with  $r > 1$ , then the  $u_j$  must belong to one of these cycles; choose this cycle to be last among the cycles of  $g$  in the current cycle of components. Starting with index 1 again for simplicity, it is labeled as  $u_1 \dots u_p, u_1 + v_1 \dots u_p + v_p, \dots, u_1 + v_{(r-2)p+1} \dots u_p + v_{(r-1)p}$ . Now  $f_{uv}(u'_p) = u_1 + v_1$  shows that  $\pm v_m$  is no longer absent from (14)–(16). We have a block of size  $(r-1)p$  on the diagonal of  $P$ , with nonzero entries  $1/f'(\tilde{z}_j)$  directly above the diagonal, and further entries  $-1/f'(\tilde{z}_p)$  in rows  $p, 2p, \dots, (r-1)p$  of the first column. After rescaling all variables appropriately, this is the companion matrix of  $\lambda^{(r-1)p} + \rho^{-1}\lambda^{(r-2)p} + \dots + \rho^{-(r-1)} = \frac{\lambda^{rp} - \rho^{-r}}{\lambda^p - \rho^{-1}}$ . Note that  $A$  contains a cyclic  $p$ -block for  $u_1, \dots, u_p$  again, but since  $A$  is not block-triangular, it need not have eigenvalues with  $\lambda^p = \rho^{-1}$ . If  $g$  has further  $r'p$ -cycles in the same components of  $\widehat{\mathbb{C}} \setminus \Gamma$ , these are treated according to the first scenario, giving  $\lambda^{r'p} = \rho^{-r'}$ ; there will be additional entries above the diagonal blocks, which do not contribute to the characteristic polynomial of  $P$ , but may prevent it from being diagonalizable. — An alternative approach to the second scenario would be to modify  $g$  isotopically so that it has a  $p$ -cycle in the disks, and to mark this cycle in addition.

So if  $f$  is not of type (2, 2, 2, 2), all eigenvalues of the extended pullback relation  $x \mapsto x'$  at the fixed point  $x^\infty$  are attracting. We shall construct a norm on  $\mathbb{C}^{|Z|-3}$ , such that the linearization satisfies  $\|x' - x^\infty\| \leq L\|x - x^\infty\|$  for some  $L < 1$ ; in a small neighborhood  $\mathcal{N}$  with  $\|x - x^\infty\| < \delta$  the pullback map is analytic and satisfies  $\|x' - x^\infty\| \leq L'\|x - x^\infty\|$  for some  $L < L' < 1$ . To define this norm, conjugate the derivative matrix to its Jordan normal form by a linear change of variables. Rescale components such that the entries 1 above the diagonal become  $\varepsilon$ , and choose  $\varepsilon > 0$  small such that the new matrix is contracting with respect to the standard Euclidean norm. The norm  $\|\cdot\|$  in the original coordinates corresponds to this Euclidean norm.

In the remaining cases 2 and 3, we will not have an analytic branch of  $x \mapsto x'$  in a neighborhood of  $x^\infty$ , but we shall construct an attracting neighborhood for the path  $x(t)$  nevertheless when  $f$  is not of type (2, 2, 2, 2). In **case 2**, suppose that  $z_m$  is identified with a critical point  $\omega$ . If  $g^k(z_{\alpha'}) = z_{\beta'}$  and  $\omega = z_{\alpha'}$ , then  $g^k(z_m)$  is identified with  $z_{\beta'}$ , and we shall redefine  $\omega = z_{\beta'}$  and  $m$  such that  $z_m$  is identified with  $\omega$ . Preimages of  $z_m$  and  $\omega$  may be marked or not, and identified or not. Define new coordinates  $(u, v, w)$  with  $w$  representing all  $x_i$ , such that  $z_i$  is identified with a critical or precritical point, and  $u, v$  describing the remaining  $x_i$  as in case 1. The marked point  $x_{\gamma'}$  normalized to 1 may be precritical but not be identified with a precritical or critical point. So the rational maps  $f_{uv}$  will not depend on  $x_m$  and its preimages; the multi-valued pullback relation  $(u, v, w) \mapsto (u', v', w')$  is such that  $u'$  and  $v'$  do not depend on  $w$ . As in case 1, we have a local analytic branch and an attracting neighborhood  $\mathcal{N}_0$  for  $(u, v) \mapsto (u', v')$ .

Now the pullback for  $x_m$  is asymptotic to  $x'_m \sim \sqrt[d]{c \cdot v_j}$  or  $x'_m \sim \sqrt[d]{c/v_j}$  when  $\omega = z_{\beta'}$  or  $\omega = z_{\alpha'}$ , respectively. The branch of the root is defined uniquely along a path, but there is no analytic branch in a neighborhood of 0 or  $\infty$ . Moreover, this expression does not seem to be attracting, but it is used for a preperiodic point here. So we only need it to be continuous in the sense that  $v_j \rightarrow 0$  implies  $\sqrt[d]{c \cdot v_j} \rightarrow 0$  or  $\sqrt[d]{c/v_j} \rightarrow \infty$  for any branch. For  $(u, v) \in \mathcal{N}_0$ ,  $x'_m$  will be in a small neighborhood of 0 or  $\infty$ , and its preimages will be in small neighborhoods of precritical points of  $f$ . The product of  $\mathcal{N}_0$  with these neighborhoods defines the attracting neighborhood  $\mathcal{N}$  for  $(u, v, w)$ . Note that the coordinates  $u$  and  $v$  converge geometrically as  $\mathcal{O}(L^t)$ , and  $w$  with  $\mathcal{O}(L^{t/d})$ , or  $\mathcal{O}(L^{t/d^2})$  if  $f^k(\infty) = 0$  or  $f^k(0) = \infty$ .

In **case 3**,  $f^k(\infty) = 0$  for some  $k \geq 1$ , and the postcritical point  $z_m = g^k(z_{\alpha'})$  is identified with  $z_{\beta'}$ . Consequently, iterates of  $z_m$  are identified with corresponding iterates of  $z_{\beta'}$ . If preimages of  $z_{\beta'}$  are identified with preimages of  $z_m$  as well, or if additional non-postcritical marked points are identified with critical or precritical points, they are labeled  $w$  and treated separately as in case 2 — these points will be ignored from now on. The pullback relation is not reducible in case 3: we have  $x'_m \sim \sqrt[d]{c \cdot v_j}$  again, and this coordinate cannot be treated separately, since it is pulled back to the critical value  $x_\alpha$ . This value appears in the parameters of  $f_{uv}$  and influences the pullback of every point. Postcritical variables  $v$  may appear directly in these parameters, if  $k = 1$  or  $f(0)$  has preperiod 1. Note that  $x'_m \sim \sqrt[d]{c \cdot v_j}$  is the only component of (12) not analytic in a neighborhood of  $x = x^\infty$  or  $(u, v) = (\tilde{z}, 0)$ .

Choose the coordinates  $(u, v)$  such that preperiodic iterates of  $z_{\beta'}$  are of type  $u$  and preperiodic iterates of  $z_m$  are of type  $u + v$ , including  $x_m = v_m$ . So  $v'_m \sim \sqrt[d]{c \cdot v_j}$  and the partial derivative  $\partial v'_m / \partial v_j \rightarrow \infty$  as a branch of the root is continued analytically along the path. In a way, the matrix  $D$  in (13) has a unique infinite entry, in block  $R$  and above the diagonal. One idea to deal with this is to use the orbifold metric of  $f$  for  $x_i \approx x_i^\infty = \tilde{z}_{D(i)}$  instead of the usual metric on  $\mathbb{C}$ . Alternatively, we may lift the path and the pullback relation to new coordinates  $(U, V)$  with  $U_i = u_i$  for all  $i$  and, e.g.,  $V_m = v_m$  but  $V_j^d = v_j$ ; this gives  $V'_m \sim \sqrt[d]{c} \cdot V_j$ , where the branch of  $\sqrt[d]{c}$  is determined by the chosen lift of a concrete path. The  $v$ -coordinates of iterates must be lifted as well, but this gives an analytic pullback relation only when  $g$  has a postcritical cycle of the same period  $p$  as  $f$ . If we are in the second scenario, simply add a  $p$ -cycle to  $g$  within the cycle of disks; this does not change the pullback of the other points in the  $x$ -coordinates, but when the new points are used as  $u$ -coordinates, this allows to estimate the  $v$ -coordinates. — In the lifted coordinates, we have attracting eigenvalues and an attracting neighborhood as in case 1.

In any case, the neighborhood  $\mathcal{N}$  can be chosen such that its preimages under different branches of the pullback relation are either contained in  $\mathcal{N}$  or disjoint from it. Since path segments are appended continuously, the given segment stays in  $\mathcal{N}$  forever and is attracted. ■

Note that when  $\Gamma$  would be replaced with a non-homotopic multicurve  $\hat{\Gamma} \in G \cdot \Gamma$ , which is grouping marked points in the same way, the new  $\tilde{g}$  may be obstructed or equivalent to a different  $f$ . On the other hand, we have not used information on the homotopy class of  $\Gamma$  in the proof of Proposition 3.12. Here the key point is the assumption on the path in item 2: for a different  $\hat{\Gamma}$ , the attracting neighborhood  $\mathcal{N}$  would be the same, but there may be no path segment of length 1 contained in it.

The same remark applies to the usual Thurston Algorithm without identifications in fact: the pullback relation will have several attracting fixed points, and one of these is chosen depending on the isotopy class of  $g$  or on the initial path segment.

**Proof of Theorem 3.11.2:** The marked point  $z_{\gamma'}$  normalized to  $x_{\gamma'} = 1$  is chosen such that together with the critical points at  $x_{\beta'} = 0$  and  $x_{\alpha'} = \infty$ , it singles out the component  $C$ . The associated embedding  $\pi_3 : \mathcal{M} \rightarrow \widehat{\mathbb{C}}^{|Z|}$  extends continuously to  $\pi_3 : \widehat{\mathcal{M}} \rightarrow \widehat{\mathbb{C}}^{|Z|}$  according to Proposition 3.3.3; on  $\mathcal{S}_{G,\Gamma}$  it is described as follows: Each  $m \in \mathcal{S}_{G,\Gamma}$  defines a noded Riemann surface; the piece corresponding to  $C$  is isomorphic to  $\widehat{\mathbb{C}}$ . The isomorphism is unique by sending specific marked points and nodes to  $0, 1, \infty$ . Now marked points in other pieces are sent to the same points as the corresponding nodes. So the fixed point  $\widehat{\tau}^C$  of  $\sigma_C = \sigma_f$  has the following property: all  $\tau \in \mathcal{S}_\Gamma$  with component  $\tau^C = \widehat{\tau}^C$  have  $\pi_3(\pi(\tau)) = x^\infty$ .

Now suppose that the choice of  $x_{\gamma'} = 1$  was made according to the restrictions from Proposition 3.12, and obtain an attracting neighborhood  $\mathcal{N}$ . Combine spherical metrics to define a metric  $d$  on  $\widehat{\mathbb{C}}^{|Z|-3} \subset \widehat{\mathbb{C}}^{|Z|}$  and choose  $\delta > 0$  such that the open ball of radius  $2\delta$  around  $x^\infty$  is contained in  $\mathcal{N}$ . Proposition 3.12 applies with the same notation of  $g, \Gamma, C$ , and  $\widehat{\tau}^C$ . We shall start by constructing a path in  $\mathcal{T} \cup \mathcal{S}_\Gamma$ ; for suitable  $0 < i < I < \infty$  it goes from  $\tau_{n_{k(I)+i}}$  to  $\sigma_g^i(\widehat{\tau}_I)$ , to  $\sigma_g^{i+1}(\widehat{\tau}_I)$ , and to  $\tau_{n_{k(I)+i+1}}$ . There is a T-ball around  $\widehat{\tau}^C$  in  $\mathcal{T}^C$ , such that all  $\tau \in \mathcal{S}_\Gamma$  with  $\tau^C$  in this ball satisfy  $d(\pi_3(\pi(\tau)), x^\infty) < \delta$ . Choose  $i$  according to Proposition 3.9.3 such that  $\sigma_C^i(\widehat{\tau}_I^C)$  and  $\sigma_C^{i+1}(\widehat{\tau}_I^C)$  belong to this ball for  $I > i$ . Since  $\widehat{\mathcal{M}}$  is compact,  $\pi_3$  is uniformly continuous. Choose  $I > i$  such that  $\varepsilon(I)$  is sufficiently small, so  $d_{WP}(m', m) < \varepsilon(I)$  implies  $d(\pi_3(m'), \pi_3(m)) < \delta$ . Now we have  $d_{WP}(\tau_{n_{k(I)+i}}, \sigma_g^i(\widehat{\tau}_I)) < \varepsilon(I)$  and  $d_{WP}(\sigma_g^{i+1}(\widehat{\tau}_I), \tau_{n_{k(I)+i+1}}) < \varepsilon(I)$  according to Proposition 3.9.2. The first and third segments of our preliminary path shall be the corresponding WP-geodesics. The middle segment from  $\sigma_g^i(\widehat{\tau}_I)$  to  $\sigma_g^{i+1}(\widehat{\tau}_I)$  shall be the product of T-geodesics in the components of  $\mathcal{S}_\Gamma$ .

So with  $n_* = n_{k(I)} + i$  we have constructed a preliminary path from  $\tau_{n_*}$  to  $\tau_{n_*+1}$  in  $\mathcal{T} \cup \mathcal{S}_\Gamma$ , such that  $d(\pi_3(\pi(\tau)), x^\infty) < 2\delta$  on this path. Since the ball is open and the path is compact, we may choose a nearby path from  $\tau_{n_*}$  to  $\tau_{n_*+1}$  in  $\mathcal{T} \cap (\pi_3 \circ \pi)^{-1}(\mathcal{N})$ . The pullback of this path interpolates  $(\tau_n)_{n \geq n_*}$  and projects to a path in  $\pi_3(\mathcal{M})$ , which stays in  $\mathcal{N}$  and converges to  $x^\infty$  according to Proposition 3.12.

In the course of these proofs, several paths were constructed and discarded to obtain convergence of the sequence  $\pi_3(\pi(\tau_n))$ . Now suppose that a path  $\tau_t$  is given from the start. Then for  $\varepsilon > 0$  we want to find  $T \geq 0$  with  $d(\pi_3(\pi(\tau_t)), x^\infty) < \varepsilon$  for  $t > T$ . This is done by applying the result for sequences to the pullback of finitely many intermediate points on the initial segment, which are chosen depending on  $\varepsilon$ , such that each smaller segment gives a change  $< \varepsilon/2$  in  $\pi_3(\mathcal{M})$ . Note again that the T-distance and thus the WP-distance stays bounded under the pullback and that  $\pi_3$  is uniformly continuous on  $\widehat{\mathcal{M}}$ .

Finally, consider the case where the marked point normalized to 1 does not satisfy the assumption of Proposition 3.12. Then we have convergence in a different normalization, and the two normalizations are related by an affine rescaling with a convergent factor. ■

**Remark 3.13 (Rate of convergence)**

1. *The attracting eigenvalues at  $x^\infty$  in configuration space were related to multi-*

pliers of  $f$  and to eigenvalues of  $D\sigma_f$  in Proposition 3.12, where  $\sigma_f$  includes both postcritical points and additional marked points. Note that similar estimates apply to collisions with critical points, and that additional marked points of  $g$  without collisions converge to marked points of  $f$  with a rate determined by multipliers of  $f$  as well. The orbifold metric [40, 34] provides uniform expansion and uniform estimates for multipliers of  $f$ , especially  $|\rho| \geq k^p$  for the multiplier of a non-postcritical  $p$ -cycle. So if  $k_f$  is a bound for the eigenvalues of  $D\sigma_f$  without additional marked points, then  $d(\pi_3(\pi(\tau_n)), x^\infty)$  asymptotically shrinks exponentially by  $\max(k^{-1}, k_f) < 1$  independently of the number of additional marked points with or without collisions.

2. This bound on eigenvalues does not directly imply uniform convergence, e.g., in the case of a formal mating  $g$  with fixed  $\psi_0$  but an arbitrary number of marked points: using a standard distance on  $\widehat{\mathbb{C}}^{|Z|^{-3}}$ , the norm of the derivative may be arbitrarily large when eigenvectors are nearly parallel. Moreover, the number of initial steps to get into a neighborhood of  $x^\infty$  may grow with  $|Z|$ . See [11] for results on uniform convergence.

3. Under the assumptions of essential equivalence according to Definition 3.10 and Theorem 3.11, the leading eigenvalue is always  $\lambda_\Gamma = 1$ . In a different situation with  $\lambda_\Gamma > 1$ , collisions shall happen faster than exponentially.

## 4 Construction and convergence of mating

We shall employ five different notions of mating: the formal mating is constructed explicitly, modified to an essential mating, and it is combinatorially equivalent to a rational map, the combinatorial mating. This is a geometric mating at the same time, since it is conjugate to the topological mating, which is defined as a quotient of the formal mating or of the polynomials in turn. While the notion of the geometric mating may be most natural, the construction best understood starts with the formal and combinatorial matings in the postcritically finite case. — Convergence properties of the formal mating are discussed in Section 4.3 in a direct application of Theorem 3.11.

### 4.1 Dynamics and combinatorics of quadratic polynomials

The dynamics of a quadratic polynomial  $f_c(z) = z^2 + c$  is understood as follows: all  $z$  with large modulus are escaping to  $\infty$  under the iteration; the non-escaping points form the filled Julia set  $\mathcal{K}_c$ . By definition, the parameter  $c$  belongs to the Mandelbrot set  $\mathcal{M}$ , if  $\mathcal{K}_c$  is connected, or equivalently, if the critical orbit does not escape. Then the Boettcher map  $\Phi_c : \widehat{\mathbb{C}} \setminus \mathcal{K}_c \rightarrow \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  maps dynamic rays  $\mathcal{R}_c(\theta)$  to straight rays with angle  $\theta$  [40, 47, 38].

When  $\theta$  is periodic or preperiodic under doubling, the landing point  $z = \gamma_c(\theta) \in \partial\mathcal{K}_c$  is periodic or preperiodic under  $f_c$  as well. In the parameter plane, parameters  $c = \gamma_{\mathcal{M}}(\theta) \in \partial\mathcal{M}$  are defined as landing points of parameter rays with rational angles. If  $\theta$  is periodic,  $c$  is the root of a unique hyperbolic component with a unique center; for that parameter, the critical orbit is periodic. Preperiodic angles give Misiurewicz parameters, for which the critical value is preperiodic. Dynamic rays landing together are important for ray connections. For parameters  $c$  in a limb



of  $\mathcal{M}$ , the fixed point  $\alpha_c$  has unique angles and a unique rotation number, while the other fixed point always satisfies  $\beta_c = \gamma_c(0)$ .

## 4.2 Formal mating and combinatorial mating

The formal mating  $g = P \sqcup Q$  of  $P(z) = z^2 + p$  and  $Q(z) = z^2 + q$  is a Thurston map, which is conjugate to  $P$  and  $Q$  on the lower and upper half-spheres, respectively. E.g., consider an odd map  $\varphi_0 : \mathbb{C} \rightarrow \mathbb{D}$  with  $\varphi_0(r \cdot e^{i\theta}) \rightarrow e^{i\theta}$  as  $r \rightarrow \infty$  and set  $\varphi_\infty(z) = 1/\varphi_0(z)$ ; then define  $g = \varphi_0 \circ P \circ \varphi_0^{-1} \cup \varphi_\infty \circ Q \circ \varphi_\infty^{-1}$ . A simple explicit choice is given by  $\varphi_0(z) = z/\sqrt{|z|^2 + 1}$ , then  $g$  will be smooth but not quasi-regular. **External rays** of  $g$  are unions of  $\varphi_0(\mathcal{R}_p(\theta))$  and  $\varphi_\infty(\mathcal{R}_q(-\theta))$  together with a point on the equator; **ray-equivalence classes** are maximal connected unions of rays and landing points in  $\varphi_0(\partial\mathcal{K}_p)$  and  $\varphi_\infty(\partial\mathcal{K}_q)$ . Their geometry is described in [26]. According to [54, 52], in the postcritically finite case there are:

- Cyclic ray connections corresponding to non-removable Lévy cycles, when the parameters  $p$  and  $q$  are in conjugate limbs of the Mandelbrot set.
- Otherwise only trees giving identifications within and between Julia sets, maybe in several steps.
- If postcritical or additional marked points are in the same ray-equivalence class, these are surrounded by removable Lévy cycles. Then an **essential mating**  $\tilde{g}$  is defined by modifying  $g$ : these trees or disks are collapsed to points and the map is modified at preimages as well, giving an unobstructed Thurston map with a smaller number of marked points [54, 52].

The Thurston algorithm for  $g$  gives a sequence of homeomorphisms  $\psi_n$  and of rational maps  $f_n$  with  $\psi_n \circ g = f_n \circ \psi_{n+1}$ . The homeomorphisms  $\psi_n$  converge up to isotopy, unless  $g$  is obstructed or of type  $(2, 2, 2, 2)$ . The following result is classical:

### Theorem 4.1 (Combinatorial mating by Rees–Shishikura–Tan)

*Suppose the polynomials  $P$  and  $Q$  are postcritically finite and the parameters are not in conjugate limbs of the Mandelbrot set. Then the formal mating  $g = P \sqcup Q$  does not have a non-removable obstruction, and the **combinatorial mating**  $f$  is obtained as follows:*

- If the formal mating  $g$  does not have a removable obstruction, then  $f$  is combinatorially equivalent to  $g$ , and the Thurston Algorithm for  $g$  converges  $f_n \rightarrow f$ .*
- If the formal mating  $g$  has a removable obstruction, then  $f$  is defined as the rational map equivalent to the essential mating  $\tilde{g}$ . The Thurston Algorithm for  $\tilde{g}$  converges  $f_n \rightarrow f$ , unless  $\tilde{g}$  has an orbifold of type  $(2, 2, 2, 2)$ .*

We may speak of “the” combinatorial mating, since Möbius conjugacy classes are avoided by assuming a normalization: the critical point 0 of  $f$  corresponds to  $P$ , the critical point  $\infty$  to  $Q$ , and 1 is the fixed point of argument 0. Different combinatorial matings might still be conjugate to each other by marking a different fixed point, or by interchanging  $P$  and  $Q$ . E.g., the combinatorial matings of  $z^2 \pm i$  with the Basilica  $z^2 - 1$  are distinct, but conjugate by a rotation of the fixed points. This ambiguity is avoided with the alternative normalization  $f(\infty) = 1$ .

**Idea of the proof:** In [54] it is shown that every obstruction of a quadratic Thurston map contains a removable Lévy cycle, or there is a “good” Lévy cycle. The curves are homotopic to periodic ray-equivalence classes. See Figure 3. Removable cycles correspond to loops around simply connected ray-equivalence classes, while cyclic ray connections indicate the presence of non-removable Lévy cycles. Then there is a good Lévy cycle corresponding to closed ray connections between the two  $\alpha$ -fixed points, which exist precisely when the parameters are in conjugate limbs. Otherwise the essential mating  $\tilde{g}$  can be defined as a branched covering, which is unobstructed. When  $\tilde{g}$  has orbifold type not  $(2, 2, 2, 2)$ , the proof of existence and uniqueness is completed with the Thurston Theorem 2.7, but the case of type  $(2, 2, 2, 2)$  is different: here the absence of obstructions for  $\tilde{g}$  is not sufficient to guarantee that there is an equivalent rational map  $f$ . The proof can be given by applying the Shishikura Algorithm to each essential mating in question, to determine the matrix of the real-affine lift and to check that the eigenvalues are not real. The cases of  $1/4 \sqcup 1/4$  and  $1/6 \sqcup 1/6$  are described in [54], five more cases are discussed in [39], and the remaining two cases are settled in [27]. ■

For degree  $d \geq 3$  analogous definitions are used, but there is no combinatorial characterization of cyclic ray connections in general. Obstructions need not contain Lévy cycles, and non-removable obstructions may exist also when there are no cyclic ray connections [53]. Moreover, the combinatorial mating will not be unique in the case of flexible Lattès maps.

The **topological mating**  $P \amalg Q$  is defined by collapsing all rational and irrational ray-equivalence classes to points. Alternatively, take the disjoint union of  $\mathcal{K}_p$  and  $\mathcal{K}_q$  and consider the equivalence relation generated by  $\gamma_p(\theta) \sim \gamma_q(-\theta)$ . By the Moore Theorem [42], we have a Hausdorff space homeomorphic to the sphere, when the ray-equivalence relation of  $P \sqcup Q$  is closed and not separating. Then the formal mating  $g = P \sqcup Q$  descends to a branched covering of the quotient space, so the topological mating  $P \amalg Q$  is a branched covering of the sphere, which is defined up to conjugation. When mating polynomials from conjugate limbs of  $\mathcal{M}$ , the topological mating does not exist because the sphere would be pinched. It may happen that there is not even a Hausdorff space; examples of nested closed ray-equivalence classes are given in [26].

Now the **geometric mating** is a rational map  $f$  topologically conjugate to the topological mating,  $f \cong P \amalg Q$ . In the postcritically finite case of mating, the following result from [52, 12] shows that every combinatorial mating from non-conjugate limbs is a geometric mating in fact. Moreover, the topological mating exists and there are no cyclic irrational ray connections either. Note that in contrast to Theorems 4.1 and 4.3, an orbifold of type  $(2, 2, 2, 2)$  does not require special considerations.

#### Theorem 4.2 (Rees–Shishikura)

*For postcritically finite quadratic polynomials  $P$  and  $Q$ , with  $p$  and  $q$  not in conjugate limbs of  $\mathcal{M}$ , consider the formal mating  $g = P \sqcup Q$  and the essential mating  $\tilde{g}$ . According to Theorem 4.1, the combinatorial mating  $f$  is combinatorially equivalent to  $\tilde{g}$ . Moreover:*

1. *There is a semi-conjugation  $\Psi_\infty$  from  $g$  to  $f$ .*
2.  *$\Psi_\infty$  maps ray-equivalence classes to points, and it is injective otherwise. So the*

topological mating is defined on a Hausdorff space homeomorphic to the sphere, and  $\Psi_\infty$  descends to a topological conjugation from the topological mating  $P \amalg Q$  to  $f$ . Now the geometric mating of  $P$  and  $Q$  exists and it is given by  $f$ .

3.  $\Psi_\infty$  is a uniform limit of homeomorphisms.

Let us look at details from the proof for later reference in Section 4.3. Note that the Thurston Theorem 2.7 and its application in Theorem 4.1 used convergence of isotopy classes  $[\psi_n]$  for an arbitrary  $\psi_0$ , but now we need convergence of maps  $\Psi_n$  for a special choice of  $\Psi_0$ .

Idea of the **proof**: 1. The simplest case concerns preperiodic  $P$  and  $Q$  without postcritical identifications. There are isotopic  $\Psi_0, \Psi_1$  with  $\Psi_0 \circ g = f \circ \Psi_1$  and a path  $\Psi_t \in [\Psi_0]$  with  $\Psi_t \circ g = f \circ \Psi_{t+1}$  for  $0 \leq t < \infty$ . Since  $f$  is uniformly expanding with respect to the orbifold metric [40, 34], the homotopic length of a segment  $\{\Psi_t(z) \mid n \leq t \leq n+1\}$  shrinks exponentially in  $n$ , uniformly in  $z$ . So  $\Psi_\infty = \lim \Psi_n$  is continuous, surjective, and a semi-conjugation.

The second case includes hyperbolic  $P$  or  $Q$ . Then the orbifold metric is more singular at periodic critical orbits, and exponential shrinking is uniform away from these only. Although  $\Psi_0$  can be chosen as a local conjugation at these cycles by employing the Böttcher conjugation, this does not guarantee  $\Psi_1 = \Psi_0$  there. In [52] the latter property is obtained by modifying  $\Psi_0$  with suitable Dehn twists. In [12],  $\Psi_0$  is left unchanged but its pullback is described locally in terms of Dehn twists.

The third case requires the construction of an essential mating  $\tilde{g}$  by collapsing a family  $Y$  of critical and postcritical ray-equivalence classes. Then we have  $\Psi_t = \tilde{\Psi}_t \circ \pi_t$ , where  $\tilde{\Psi}_t$  is a pseudo-isotopy for the essential mating. For  $n \leq t < n+1$ ,  $\pi_t$  collapses the ray-equivalence classes in  $g^{-n}(Y)$  to points independent of  $t$ . So restricted to  $n \leq t \leq n+1$ ,  $\Psi_t$  is a pseudo-isotopy outside of finitely many ray-trees.

2. The homotopic length of subarcs of rays shrinks exponentially, at least away from precritical and postcritical classes. Moreover, any ray-equivalence class has a finite number of rays, so its image under  $\Psi_t$  shrinks to a point. It is quite involved to show that distinct classes are mapped to distinct points by  $\Psi_\infty$  [52].

3. When  $\tilde{g} = g$ ,  $\Psi_\infty$  is the end of the pseudo-isotopy  $\Psi_t$ . Otherwise both  $\tilde{\Psi}_n$  converges to a continuous map and  $\Psi_n$  converges to  $\Psi_\infty$ , but  $\Psi_n = \tilde{\Psi}_n \circ \pi_n$  is not a homeomorphism. Now the projections  $\pi_n$  are approximated by homeomorphisms, observing that ray-equivalence classes are collapsed successively. ■

Now  $\gamma_f(\theta) = \Psi \circ \varphi_0 \circ \gamma_p(\theta) = \Psi \circ \varphi_\infty \circ \gamma_q(-\theta)$  is a semi-conjugation from the angle doubling map on  $\mathbb{R}/\mathbb{Z}$  to  $f$  on its Julia set, which is not injective but can be approximated by embeddings; it gives a Peano curve when both  $P$  and  $Q$  are preperiodic. It maps the Brolin measure on  $\mathbb{R}/\mathbb{Z}$  to the Lyubich measure on  $\hat{C}$ . A tiling is obtained from  $T = \gamma_f([0, 1/2])$  as well: then the Julia set is  $T \cup (-T)$  and  $T \cap (-T)$  is the image of the spines intersected with the polynomial Julia sets [39]. In general this gives no finite subdivision rule. Alternative constructions with a pseudo-equator [36] or Hubbard trees [57] are possible in certain cases.

### 4.3 Convergence properties of the formal mating

According to the discussion of Theorem 3.11, there is no need to correct a removable obstruction by identifying marked points manually: it will be removed automatically during the iteration of the unmodified Thurston Algorithm, in the sense that several marked points have the same limit, at least in the non-(2, 2, 2, 2) orbifold case. Then  $[\psi_n]$  diverges in Teichmüller space, but the images of marked points and the rational maps  $f_n$  converge. Now the Thurston Algorithm can be implemented for the formal mating without dealing with the combinatorics and topology of postcritical ray-equivalence classes; the essential mating is used only as a step in the proof, but not in the actual pullback. See Section 5 for a discussion of slow mating. Actually, the same technique gives identifications for all repelling periodic and preperiodic points by marking them in addition. As conjectured in [7, 10], e.g., the proof is based on the Selinger extension to augmented Teichmüller space in Section 3.

#### Theorem 4.3 (Convergence of maps & rational ray-equivalence classes)

*Consider the Thurston Algorithm  $[\psi_n]$  with any initial  $\psi_0$  for the formal mating  $g = P \sqcup Q$  of postcritically finite quadratic polynomials  $P$  and  $Q$ , with  $p$  and  $q$  not in conjugate limbs of the Mandelbrot set. Moreover, assume that the combinatorial mating  $f$  has an orbifold not of type (2, 2, 2, 2).*

1. *If the formal mating  $g$  has removable obstructions, it is essentially equivalent to the combinatorial mating  $f$ . The rational maps  $f_n$  from the unmodified Thurston Algorithm converge to  $f$ . The images of marked points of  $g$  collide according to their ray-equivalence classes under the iteration, and converge to marked points of  $f$ .*
2. *In both cases, when  $g$  is combinatorially equivalent or essentially equivalent to  $f$ , consider the evolution of any periodic or preperiodic point  $z$ , which corresponds to a point in  $\partial\mathcal{K}_p$  or  $\partial\mathcal{K}_q$ : then  $x_n = \psi_n(z)$  converges to a periodic or preperiodic point of  $f$ . Different points are identified in the limit, if and only if they belong to the same ray-equivalence class.*

The second item is motivated by the videos of moving Julia sets, which are computed from the slow mating algorithm and meant to represent equipotential gluing [11]; it does not make sense when the formal mating is considered only up to isotopy with respect to postcritical points. There are two ways of understanding the statement in the context of the Thurston Algorithm:

- The formal mating  $g$  is defined such that it is topologically conjugate to  $P$  on the lower hemisphere and to  $Q$  on the upper hemisphere. So there are subsets of the sphere corresponding to the Julia sets  $\mathcal{K}_p$  and  $\mathcal{K}_q$  and points corresponding to periodic and preperiodic points of  $P$  and  $Q$ . Pick a homeomorphism  $\psi_0$  and consider its lifts with  $f_n \circ \psi_{n+1} = \psi_n \circ g$  in a suitable normalization. When  $\tilde{g} = g$ , the Thurston Theorem 2.7 shows that the homeomorphisms  $\psi_n$  converge up to homotopy with respect to the postcritical set of  $g$ , and  $\psi_n(z)$  converges when  $z$  is a marked point. So here the latter statement is extended to other points  $z$  for the same sequence  $\psi_n$ , not for any homotopic sequence.
- A finite number of these periodic or preperiodic points of  $g$  may be marked in addition, giving a new pullback map on a higher-dimensional Teichmüller space. Then  $\psi_n$  may be considered up to homotopy with respect to the marked

set  $Z$ . If there are no collisions, the Thurston Theorem 2.7 gives convergence immediately, but Theorem 3.11 is needed in general.

**Proof:** Assume a normalization with critical points at 0 and  $\infty$ , and  $1 = f_n(\infty)$  or the fixed point on the 0-ray is at 1. Actually, in the latter case we may mark  $z = 1$  and the two  $\beta$ -fixed points in addition; so use the former normalization in the proof of item 1 and treat the second normalization as a special case of item 2.

1. According to [54], all obstructions of  $g$  contain removable Lévy cycles, which consist of loops around periodic ray-equivalence classes with at least two marked points. A simple obstruction  $\Gamma$  is obtained by adding all essential preimages, which are loops around preperiodic ray-equivalence classes containing at least two critical or postcritical points. Define the essential mating  $\tilde{g}$  by identifying all of these ray-equivalence classes, or alternatively, all disks bounded by  $\gamma \in \Gamma$ , to points. Then modify the map in neighborhoods of preimages containing at most one marked point as well. This is done without destroying the orbit of a single marked point within a disk; see Sections 3.2 and 3.5. Note that the original definition [54, 52] may involve collapsing a larger number of ray-equivalence classes with a single critical or postcritical point, but all possible choices of  $\tilde{g}$  are combinatorially equivalent. Now  $g$ ,  $\Gamma$ , and  $\tilde{g}$  satisfy the assumptions of Definition 3.10:

- Again by [54],  $\tilde{g}$  is unobstructed. Since its orbifold is not of type  $(2, 2, 2, 2)$ , there is an equivalent rational map  $f$ , which defines the combinatorial mating and the geometric mating in fact.
- The critical points of  $g$  are not identified in  $\tilde{g}$ , because then  $\tilde{g}$  would not be defined properly as a branched covering of degree 2; this happens only when there are non-removable obstructions and the parameters are in conjugate limbs. No critical point is identified with 1 either, using a normalization different from  $f(\infty) = 1$  when  $f(\infty) = 0$ .
- Loops bounding small tubular neighborhoods of disjoint simply connected ray-equivalence classes define disjoint disks, so  $\Gamma$  is not nested.
- When a ray-equivalence class is not mapped homeomorphically, it contains a critical point of  $P$  or  $Q$ , so it is preperiodic: periodic critical points are superattracting and not accessible by external rays. — Note that the first-return map of the disk around a periodic ray-equivalence class gives a homeomorphism of the corresponding piece, which is always finite-order and not pseudo-Anosov [23], since the postcritical points are connected by a tree mapped to itself.

So the formal mating  $g$  is essentially equivalent to  $f$ ,  $\Gamma$  is the canonical obstruction, and Theorem 3.11 gives convergence of  $f_n \rightarrow f$ , and of colliding postcritical points as well. When  $f$  is of type  $(2, 2, 2, 2)$ , this statement is wrong in general [27].

2. Assume again that a postcritical point is normalized to 1, which is not in the same ray equivalence class as 0 or  $\infty$ , and the fixed point on the equator is marked in addition. Its convergence is obtained together with all marked points, and the normalization can be changed afterward to 1 by an affine rescaling with a convergent factor. — Given a finite number of periodic or preperiodic points of  $P$  and  $Q$ , add all of their images and all ray-equivalent points, and consider the corresponding points

in  $\varphi_0(\mathcal{K}_p)$  and  $\varphi_\infty(\mathcal{K}_q)$  together with the corresponding points on the equator of  $g$ . Denote the union of postcritical ray-equivalence classes by  $X$  and the additional classes by  $Y$ , set  $X' = g^{-1}(X) \setminus X$  and  $Y' = g^{-1}(Y) \setminus Y$ . Now we have  $X \cap Y = \emptyset$  and  $X' \cap Y' = \emptyset$ , but we may have  $X' \cap Y \neq \emptyset$ .

So there are finitely many disjoint ray-equivalence classes to consider. Each of these is a tree, since otherwise the topological and geometric matings would not exist. The essential mating  $\tilde{g}$  shall be defined by collapsing  $X$  and modifying the new map in a small neighborhood of  $X'$ . The essential map  $\hat{g}$  for the larger set of marked points is defined by collapsing  $X \cup Y$  and modification in a neighborhood of  $X' \cup Y'$ . Denote by  $\Gamma$  the union of loops around the ray-equivalence trees in  $X \cup Y$ ; it is a simple obstruction again.

When a homeomorphism  $\psi_0$  is chosen and the Thurston pullback  $f_n \circ \psi_{n+1} = \psi_n \circ g$  is applied, this gives the same homeomorphisms  $\psi_n$  and rational maps  $f_n$  as in item 1; these maps do not depend on the additional marked points, since the three normalized points are critical or postcritical. So the question is, do the homeomorphisms converge on the additional marked points, which follows when they converge in the larger Teichmüller space, i.e. up to homotopy with respect to the larger marked set. To apply Theorem 3.11 we only need to show that  $\hat{g}$  is unobstructed and equivalent to  $f$ . Otherwise for the larger set of marked points, the canonical obstruction of  $g$  would contain a loop around several disks of  $\Gamma$  or marked points of  $g$ .

- In the case without postcritical identifications in the formal mating, so  $\tilde{g} = g$ , consider  $\Psi_n$  from the proof of the Rees–Shishikura Theorem 4.2, which is defined by pulling back a specific homeomorphism  $\Psi_0$ . Then  $\Psi_n \rightarrow \Psi_\infty$ , which is a semi-conjugation mapping different ray-equivalence classes to different points. So the convergence claim is true for  $\psi_0 = \Psi_0$  and  $\hat{g}$  is unobstructed.
- When  $\tilde{g} \neq g$ , we have  $\Psi_n = \tilde{\Psi}_n \circ \pi_n \rightarrow \Psi_\infty$ , but  $\Psi_0$  is not a homeomorphism. So consider  $\tilde{\Psi}_n$  instead, which are defined by a pullback with the essential mating  $\tilde{g}$ . Since the essential map  $\hat{g}$  is defined by collapsing ray-equivalence classes of  $g$  in  $X \cup Y$  and modification around  $X' \cup Y'$ , an equivalent map can be defined as a component map from  $\tilde{g}$  as well. The limit of  $\tilde{\Psi}_n$  exists according to [12] and if  $\hat{g}$  was obstructed, then  $\Psi_\infty$  would map different ray-equivalence classes to the same point.

Now Theorem 3.11 applies and gives convergence for any initial  $\psi_0$ , in particular for slow mating and for equipotential gluing. See Remark 3.13 and [11] for questions of uniform convergence with respect to an arbitrary number of additional marked points. ■

There are a few related ways to describe, which periodic or preperiodic points of  $g$  converge to which point of  $f$ :

- According to the Rees–Shishikura Theorem 4.2, there is a semi-conjugation  $\Psi = \Psi_\infty$  from  $g$  to  $f$ , which maps each ray-equivalence class to a unique point.
- Then  $\Psi \circ \varphi_0$  and  $\Psi \circ \varphi_\infty$  are partial semi-conjugations from  $P$  on  $\mathcal{K}_p$  or  $Q$  on  $\mathcal{K}_q$  to restrictions of  $f$ .

- $\gamma_f(\theta) = \Psi \circ \varphi_0 \circ \gamma_p(\theta) = \Psi \circ \varphi_\infty \circ \gamma_q(-\theta)$  is a semi-conjugation from the angle doubling map on  $\mathbb{R}/\mathbb{Z}$  to  $f$  on its Julia set.

## 5 Implementation of slow mating

Equipotential gluing was defined by Milnor [37]. For any radius  $1 < R < \infty$  the equipotential lines of potential  $\log R$ ,  $|\Phi_p(z)| = R$  and  $|\Phi_q(z)| = R$ , are glued to form a sphere  $S_R$  with conformal images of  $\mathcal{K}_p$  and  $\mathcal{K}_q$ . There are associated quadratic rational maps  $f_R : S_{\sqrt{R}} \rightarrow S_R$ , and it is expected that  $f_R$  converges to the conformal mating  $f \cong P \amalg Q$  as  $R \rightarrow 1$ .

Slow mating shall denote any implementation of the Thurston Algorithm for the formal mating  $g = P \sqcup Q$  of postcritically finite polynomials, which is based on a pullback of a path in moduli space according to Section 2.3. A particular initialization is given below; it is assumed to approximate equipotential gluing when the initial radius  $R_1$  is large, and the limit  $t \rightarrow \infty$  corresponds to  $R \rightarrow 1$ . Buff and Chéritat have introduced slow mating to make movies of equipotential gluing, where the images of  $\mathcal{K}_p$  and  $\mathcal{K}_q$  are moving and the identification of boundaries is visible as a process. See the examples on the web pages [www.math.univ-toulouse.fr/~cheritat/](http://www.math.univ-toulouse.fr/~cheritat/) and on [www.mndynamics.com](http://www.mndynamics.com).

Recall that the formal mating  $g = P \sqcup Q$  is defined as  $g = \varphi_0 \circ P \circ \varphi_0^{-1}$  on the lower half-sphere  $|z| < 1$ , and by  $g = \varphi_\infty \circ Q \circ \varphi_\infty^{-1}$  on the upper half-sphere  $|z| > 1$ . Denote the critical orbits by  $P : 0 \Rightarrow p = p_1 \rightarrow p_2 \rightarrow \dots$  and  $Q : 0 \Rightarrow q = q_1 \rightarrow q_2 \rightarrow \dots$ ; when these are finite, the Thurston Algorithm of the formal mating shall be implemented with a path in moduli space according to Section 2.3. The homeomorphisms  $\psi_0$  and  $\psi_1$  are chosen such that  $x_i(t) = \psi_t(\varphi_0(p_i))$  satisfies  $x_i(0) \approx p_i/R_1^2$  and  $x_i(1) \approx p_i/R_1$  for some initial radius  $R_1 \gg 2$ ; analogously we require that  $y_i(t) = \psi_t(\varphi_\infty(q_i))$  satisfies  $y_i(0) \approx R_1^2/q_i$  and  $y_i(1) \approx R_1/q_i$ . There are two reasons for this: first, it is considered as an approximation of equipotential gluing. Second, we will obtain simple explicit formulas, which allow to check that there is a corresponding path in Teichmüller space. Now  $\psi_1$  shall be the pullback of  $\psi_0$ , so  $f_0 \circ (\psi_1 \circ \varphi_0) = (\psi_0 \circ \varphi_0) \circ P$  and  $f_0 \circ (\psi_1 \circ \varphi_\infty) = (\psi_0 \circ \varphi_\infty) \circ Q$  for the same quadratic rational map  $f_0$  with  $f_0(z) \approx z^2$ .

Interpolate the radius as  $\log(R_t) = 2^{1-t} \log(R_1)$  for  $0 \leq t < \infty$ . For the initial path segment  $0 \leq t \leq 1$  we cannot take  $x_i(t) = p_i/R_t$  and  $y_i(t) = R_t/q_i$ , since there would be no quadratic rational map  $f_0$  sending  $x_{i-1}(1)$  and  $y_{i-1}(1)$  to  $x_i(0)$  and  $y_i(0)$ , respectively. We might focus on  $t = 0$ , define  $x_i(0)$  and  $y_i(0)$  explicitly and determine  $f_0$ ,  $x_i(1)$ , and  $y_i(1)$  accordingly, but to avoid computing and estimating square-roots, we shall take the opposite direction: choose the simplest formula at time  $t = 1$ . Since  $z = 1$  is assumed to be fixed, consider

$$x_i(1) = \frac{p_i}{R_1} \quad , \quad y_i(1) = \frac{R_1}{q_i} \quad , \quad \text{and} \quad f_0(z) = \frac{1 + q/R_1^2}{1 + p/R_1^2} \cdot \frac{z^2 + p/R_1^2}{1 + q/R_1^2 z^2} \quad . \quad (17)$$

From  $p_{i-1}^2 = p_i - p$  and  $q_{i-1}^2 = q_i - q$ , we obtain at time  $t = 0$

$$x_i(0) = f_0(x_{i-1}(1)) = \frac{1 + q/R_1^2}{1 + p/R_1^2} \cdot \frac{p_i/R_1^2}{1 + q/R_1^4 (p_i - p)} \approx \frac{p_i}{R_1^2} \quad \text{and} \quad (18)$$

$$y_i(0) = f_0(y_{i-1}(1)) = \frac{1 + q/R_1^2}{1 + p/R_1^2} \cdot \frac{R_1^2(1 + p/R_1^4(q_i - q))}{q_i} \approx \frac{R_1^2}{q_i}. \quad (19)$$

Note that in the periodic case, this choice of  $f_0$  is optimal, because the periodic critical points stay at 0 and  $\infty$ . In the preperiodic case, there are opposite marked points for  $t \geq 1$ , which are no longer opposite for  $0 \leq t < 1$ . This is violating our usual normalization, and in a sense it is moving the unmarked critical points, but it does not really matter for  $t \geq 1$ . Now we shall use the following interpolation  $x_i(t)$  and  $y_i(t)$ :

### Initialization 5.1 (Slow mating)

For postcritically finite polynomials  $P$  and  $Q$  with critical orbits  $(p_i)$  and  $(q_i)$ , the unmodified Thurston Algorithm for the formal mating  $g = P \sqcup Q$  can be implemented with a path in moduli space as follows: Fix  $R_1 \geq 5$  and interpolate the radius as  $\log(R_t) = 2^{1-t} \log(R_1)$  for  $0 \leq t \leq 1$ . Set

$$x_i(t) = \frac{1 + (1-t)q/R_1^2}{1 + (1-t)p/R_1^2} \cdot \frac{p_i/R_t}{1 + (1-t)q/R_1^4(p_i - p)} \approx \frac{p_i}{R_t} \quad \text{and} \quad (20)$$

$$y_i(t) = \frac{1 + (1-t)q/R_1^2}{1 + (1-t)p/R_1^2} \cdot \frac{R_t(1 + (1-t)p/R_1^4(q_i - q))}{q_i} \approx \frac{R_t}{q_i}. \quad (21)$$

The initial path for  $0 \leq t \leq 1$  can be pulled back uniquely for  $1 < t < \infty$ , choosing the sign in (22) by continuity. For the homeomorphism  $\psi_0$  from the proof, the marked points  $\pi(\sigma_g^n([\psi_0]))$  are given by  $x_i(n)$  and  $y_i(n)$  in the usual normalization of 0, 1,  $\infty$ .

Actually, when  $R_1 \geq 10^{10}$ , this initialization is the same as  $x_i(t) = p_i/R_t$  and  $y_i(t) = R_t/q_i$  in 8-byte precision. — The rational maps  $f_t$  with  $f_t \circ \psi_{t+1} = \psi_t \circ g$  are obtained from the critical values  $x_1(t)$  and  $y_1(t)$ . Now the pullback is computed as follows (read  $x$  or  $y$  for  $z$ ):

$$z_i(t+1) = \pm \sqrt{\frac{1 - y_1(t)}{1 - x_1(t)} \cdot \frac{z_{i+1}(t) - x_1(t)}{z_{i+1}(t) - y_1(t)}} \quad \text{for } t \geq 0. \quad (22)$$

Note that the first factor is dropped here and above for self-matings with  $p = q$ . The formulas of initialization are simplified in general, when the alternative normalization  $f_t(\infty) = 1$  instead of  $f_t(1) = 1$  is used. See the Examples 2.5 and 2.6.

**Proof:** For  $0 \leq t \leq 1$  we need a path of homeomorphisms  $\psi_t$ . In a neighborhood of the filled Julia sets  $\varphi_0(\mathcal{K}_p)$  and  $\varphi_\infty(\mathcal{K}_q)$ , it is defined in terms of the following Möbius transformations, which are chosen such that marked points are mapped according to the given initialization. For suitable discs within  $|z| \leq 4$  set

$$\psi_t(\varphi_0(z)) = \frac{1 + (1-t)q/R_1^2}{1 + (1-t)p/R_1^2} \cdot \frac{z/R_t}{1 + (1-t)q/R_1^4(z - p)} \approx \frac{z}{R_t} \quad \text{and} \quad (23)$$

$$\psi_t(\varphi_\infty(z)) = \frac{1 + (1-t)q/R_1^2}{1 + (1-t)p/R_1^2} \cdot \frac{R_t(1 + (1-t)p/R_1^4(z - q))}{z} \approx \frac{R_t}{z}. \quad (24)$$

Between these two discs, the maps are interpolated to obtain homeomorphisms of spheres. We do not want to introduce a twist accidentally, so we shall assume that



$\psi_t$  is close to the identity at the equator, and that it maps the positive real axis close to itself. All of this makes sense when  $R_1 \geq 5$ , since  $|\psi_t(\varphi_0(z))| < 1$  and  $|\psi_t(\varphi_\infty(z))| > 1$  for  $|z| \leq 4$ , and since the maps are close to  $z/R_t$  and  $R_t/z$  there.

According to Proposition 2.4, we need to check  $f_0 \circ \psi_1 = \psi_0 \circ g$ . Then our initialization is the projection of a suitable initial path in Teichmüller space for  $0 \leq t \leq 1$ , and the pullback in moduli space is the projection of the desired path in Teichmüller space, which is interpolating  $\sigma_g^n([\psi_0])$  for  $n \in \mathbb{N}$ .

Our choices are compatible such that  $f_0 \circ (\psi_1 \circ \varphi_0) = (\psi_0 \circ \varphi_0) \circ P$  and  $f_0 \circ (\psi_1 \circ \varphi_\infty) = (\psi_0 \circ \varphi_\infty) \circ Q$  in neighborhoods of the filled Julia sets, so  $f_0 \circ \psi_1 = \psi_0 \circ g$  there. If  $\varphi_0$  was extended arbitrarily to the annulus, it might have an additional twist compared to the correct pullback. By looking at preimages of the positive real axis again, we see that this is not the case, so we may assume that  $\psi_0$  and  $\psi_t$  are defined such that  $f_0 \circ \psi_1 = \psi_0 \circ g$  globally, and  $\sigma_g([\psi_0]) = [\psi_1]$ . Note again that  $\psi_t$  is not odd for  $0 \leq t < 1$ , but the pullback will be odd for  $1 \leq t < \infty$ . ■

So in any case of postcritically finite polynomials  $P$  and  $Q$ , there is a simple numerical method to compute the pullback of a suitable homeomorphism, or more precisely, the projection  $\pi(\sigma_g^n([\psi_0]))$  to moduli space. The discretization of a path segment shall be discussed in [28, 29]. The relation of the slow mating algorithm to equipotential gluing and the visual representation will be explored further in [11]. Note that the algorithm itself is quite fast, if you only compute the maps and not the Julia sets, and the movies slow it down to illustrate the process.

When the formal mating  $g$  is unobstructed, then  $f_t$  converges to the geometric mating  $f \cong P \amalg Q$ , since the slow mating algorithm is just an explicit method to compute the pullback. When there are removable obstructions, the maps converge as well according to Theorem 4.3, at least if the orbifold of  $\tilde{g}$  or  $f$  is not of type  $(2, 2, 2, 2)$ . Marked points of  $g$  are identified automatically in the limit, and there is no need to encode the topology of postcritical ray-equivalence classes to define the pullback of the essential mating  $\tilde{g}$  instead.

However, the pullback  $\pi(\sigma_g^n([\psi_0]))$  will diverge in general when  $\tilde{g}$  has an orbifold of type  $(2, 2, 2, 2)$ . See Section 5 in [27]. Probably the path accumulates on a subset of the center manifold, which is disjoint from the fixed point, but there are other possibilities when the path intersects the stable manifold. — Even when the formal mating  $g$  has non-removable obstructions, the pullback can be computed from the slow mating algorithm, but it will diverge in a different way; e.g.,  $f_t$  may converge to a constant map.

## 6 Captures and matings

Captures and precaptures are ways to construct a Thurston map by shifting a critical value to a preperiodic point; we shall see that precaptures are related to matings with preperiodic polynomials in fact.

## 6.1 Hyperbolic and repelling-preperiodic captures

These constructions rely on the concept of shifting or pushing a point from  $a$  to  $b$  along an arc  $C$ . This means that a homeomorphism  $\varphi$  is chosen, which is the identity outside off a tubular neighborhood of  $C$ , and such that  $\varphi(a) = b$ . We may assume that an unspecified point close to  $a$  is mapped to  $a$  and  $b$  is mapped to an arbitrary point nearby.

### Proposition 6.1 (and definition)

Suppose  $P$  is a postcritically finite quadratic polynomial and  $z_1 \in \mathcal{K}_p$  is preperiodic and not postcritical. Let the new postcritical set be  $P_g = P_P \cup \{P^n(z_1) \mid n \geq 0\}$ . Consider an arc  $C$  from  $\infty$  to  $z_1$  not meeting another point in  $P_g$  and choose a homeomorphism  $\varphi$  shifting  $\infty$  to  $z_1$  along  $C$ , which is the identity outside off a sufficiently small neighborhood of  $C$ . Then:

- $g = \varphi \circ P$  is well-defined as a quadratic Thurston map with postcritical set  $P_g$ . It is a **capture** if  $z_1$  is eventually attracting and a **precapture** in the repelling case.
- The combinatorial equivalence class of  $g$  depends only on the homotopy class of the arc  $C$ .

**Proof:** By construction,  $g$  is a postcritically finite branched cover, when the neighborhood of  $C$  does not include any postcritical point except  $z_1$ . Note that the preimages of  $z_1$  under  $P$  are mapped to some arbitrary point by  $g$ , so if  $z_1$  was periodic or postcritical,  $g$  would not be well-defined. Finally, if we have two different homeomorphisms  $\varphi$  and  $\varphi'$  along the same curve or along two homotopic curves, then  $g' = (\varphi' \circ \varphi^{-1}) \circ g$  and the homeomorphism  $\varphi' \circ \varphi^{-1}$  is isotopic to the identity, since the appended path  $C' \cdot C^{-1}$  is contractible relative to  $P_g \setminus \{z_1\}$ . ■

Consider the following applications and possible generalizations:

- If a capture  $g = \varphi \circ P$  is combinatorially equivalent to a rational map  $f$ , this gives a hyperbolic map of capture type. Let us say that  $f$  is a **Wittner capture**, if the capture path  $C$  is homotopic to a rational external ray followed by an internal ray of  $P$ ; this construction is due to Ben Wittner [59] and Mary Rees [45]. Note that Rees denotes only Wittner captures as captures, while general captures are called maps of type III. Maps of this type are never matings, but they may have a representation as an anti-mating [28].
- Precaptures along external rays are related to matings in the following Section 6.2.
- Precaptures apply not only to polynomials  $P$ , but to rational maps in general as long as the other critical orbits are finite. This construction provides a finite **regluing** followed by a possible combinatorial equivalence. In a more general situation, a countable regluing is followed by a semi-conjugation [56, 32].
- **Recapture** means that the finite critical value  $P(0)$  is shifted to a preimage of 0, resulting in a Thurston map equivalent to a hyperbolic polynomial. Relations to internal addresses and to Dehn twisted maps are discussed in [29].

### Initialization 6.2 (Captures and precaptures)

Consider a capture or precapture  $g = \varphi \circ P$  according to Proposition 6.1. Then the Thurston Algorithm is implemented by pulling back a path in moduli space, which is initialized as follows: normalize  $P$  such that the critical points are  $0, \infty$  and another point in  $P_g \setminus \{z_1\}$  is  $1$ . For  $0 \leq t \leq 1$ ,  $x_1(t)$  moves from  $\infty$  to  $z_1$  along  $C$ , while all of the other marked points stay fixed.

Under a non-conjugate-limbs condition, Wittner captures are unobstructed [45] and precaptures along external rays have only obstructions satisfying the assumptions of Theorem 3.11; see below. So the sequence of rational maps converges to a rational map  $f$ , unless the orbifold of  $f$  is of type  $(2, 2, 2, 2)$ : then the sequence does not converge in general, but it might converge for a special choice of  $C$ .

**Proof:** Note that when the preperiod of  $z_1$  is one, the corresponding periodic point satisfies  $\psi_t(-z_1) = -x_1(t)$  only for  $t \geq 1$ . Since  $\varphi^{-1} \circ g = P \circ \text{Id}$  and  $P$  is holomorphic, we have  $[\text{Id}] = \sigma_g([\varphi^{-1}])$  and we may initialize the Thurston Algorithm with a path  $\psi_t$  from  $\psi_0 = \varphi^{-1}$  to  $\psi_1 = \text{Id}$ . Now  $\varphi^{\pm 1}$  is the identity outside off a small neighborhood of  $C$ , so  $\psi_t$  can be chosen such that it moves  $x_1(t) = \psi_t(z_1)$  from  $\varphi^{-1}(z_1) = \infty$  to  $z_1$  along  $C$ , and leaves the other marked points untouched. By Proposition 2.4 the projection from  $\mathcal{T}$  to  $\mathcal{M}$  defines a suitable initialization to compute the Thurston pullback  $\pi(\sigma_g^n)$  from an explicit pullback in moduli space. ■

To illustrate the process of slow capture or precapture, we may also define a sequence or path of images  $\psi_t(\mathcal{K}_p)$  of the filled Julia set, which is constant  $\mathcal{K}_p$  for  $0 \leq t \leq 1$ . It will show more and more identifications happening by a piecewise pseudo-isotopy. See also the videos on [www.mndynamics.com](http://www.mndynamics.com). A similar initialization is used for Dehn twisted maps; see [2] and the Examples 3.1 and 3.7 in [27].

## 6.2 Precaptures and matings

The representation of matings by precaptures along external rays is motivated by remarks in [32, 46]. In the former paper, the boundary of a capture component in  $V_n$  is described by matings, which are related to the postcritically finite map of capture type by regluing. This means that the critical value is shifted from  $\infty$  along an external ray followed by an internal ray, and then moved back along an internal ray. So can the mating be constructed by shifting the critical value directly from  $\infty$  to  $z_1 = \gamma_p(\theta)$  along the external ray  $\mathcal{R}_p(\theta)$ ? This is true in general when  $z_1$  is preperiodic, not only when it is on the boundary of a hyperbolic component, but we shall not discuss postcritically infinite maps here.

### Theorem 6.3 (Matings as precaptures, following Rees)

Suppose  $P$  is postcritically finite and  $\theta$  is preperiodic, such that  $q = \gamma_M(-\theta)$  is not in the conjugate limb and  $z_1 = \gamma_p(\theta) \in \partial\mathcal{K}_p$  is not postcritical. Then the precapture  $g_\theta = \varphi_\theta \circ P$  along  $\mathcal{R}_p(\theta)$  is combinatorially equivalent or essentially equivalent to the geometric mating  $f$  defined by  $P \amalg Q$ .

So if  $P \amalg Q$  is not of type  $(2, 2, 2, 2)$ , any implementation of the Thurston pullback for  $g_\theta$  gives a converging sequence of rational maps; e.g., Initialization 6.2 applies. The normalization  $\beta_p = 1$  ensures  $f(1) = 1$ . Note that the precapture does not work

if both  $P$  and  $Q$  are hyperbolic; then there is an alternative construction with two paths [45]. When only one of the two polynomials is hyperbolic, then either  $P \amalg Q$  or  $Q \amalg P$  is a precapture. And when both are critically preperiodic, then both  $P \amalg Q$  and  $Q \amalg P$  are precaptures, unless a critical point is iterated to the other critical point: then  $\infty$  shall be iterated to 0. — By choosing precaptures along homotopic external rays, examples of shared matings are obtained in [26].

Recall the notation  $g$  and  $\tilde{g}$  for the formal mating and the essential mating; we shall see below that there is an essential precapture  $\tilde{g}_\theta$  as well. Before showing  $\tilde{g}_\theta \sim \tilde{g}$  let us consider a few examples, to see how identifications happen and why they may happen in different ways for  $g$  and  $g_\theta$ :

- When  $g = 9/56 \sqcup 1/4$ , so  $\theta = 3/4$ , there are no postcritical identifications:  $\tilde{g} = g$  and  $\tilde{g}_\theta = g_\theta$ . The precapture can be constructed from the formal mating by shifting all postcritical points in  $\varphi_\infty(\mathcal{K}_q)$  to  $\varphi_0(\mathcal{K}_p)$  along external rays, so  $g_\theta$  and  $g$  are combinatorially equivalent.
- In reverse order we have  $\tilde{g} = g = 1/4 \sqcup 9/56$  again, but  $\tilde{g}_\theta \neq g_\theta$  for  $\theta = 47/56$  and  $p = \gamma_M(1/4)$ . Now  $g_\theta(\infty)$  has preperiod and period three, but  $\tilde{g}_\theta(\infty)$  has period one. The shift  $\varphi_\theta$  creates a subset of the lamination with angle  $\theta$  in the exterior of  $\mathcal{K}_p$ , so there is a triangle connecting  $3/7, 5/7, 6/7$  with a homotopic preimage under  $g_\theta$ ; pinching the surrounding Lévy-cycle gives  $\tilde{g}_\theta$ .
- The converse happens for  $g = 1/4 \sqcup 3/14$ , so  $p = \gamma_M(1/4)$  and  $\theta = 11/14$ . Now both  $q = \gamma_M(3/14)$  and  $g(\infty)$  have preperiod one and period three, while  $\tilde{g} \neq g$  has period one. But this identification is immediate in the precapture  $g_\theta = \tilde{g}_\theta$ , since  $z_1 = -\alpha_p$ .
- Both phenomena happen at the same time for  $g = 3/14 \sqcup 3/14$ , so  $\theta = 11/14$ . In  $g_\theta$  the 3-cycle of  $P$  is collapsed by a triangle in the exterior, while the 3-cycle of  $Q$  is identified with  $\alpha_p$  immediately. We have  $\tilde{g}_\theta \neq g_\theta \not\sim g \neq \tilde{g}$ .

For longer ray connections, there may be a similar splitting of branch points and similar immediate identifications, but otherwise the precapture can be understood in terms of the same ray-equivalence classes as the formal mating:

**Proof of Theorem 6.3:** Denote by  $X$  the union of all postcritical ray-equivalence classes of the formal mating  $g = P \amalg Q$ . Define another Thurston map  $g^\theta$  by shifting the critical value  $\varphi_\infty(q)$  to  $\varphi_0(z_1)$  along  $\mathcal{R}_\theta$ , without modifying  $g$  on  $X$ . Consider the extended Hubbard tree  $T_p \subset \mathcal{K}_p$ , which consists of regular arcs connecting the postcritical points of  $g_\theta$ . Then  $g_\theta : T'_p \rightarrow T_p$ , where  $T'_p = T_p$  except for a slight detour at  $P^{-1}(0)$ . We may assume that  $g^\theta \circ \varphi_0 = \varphi_0 \circ g_\theta$  in a neighborhood of  $T_p$ . So the two maps are combinatorially equivalent, even if we mark the critical point  $\infty$  in addition, since all other marked points are contained in  $T_p$  and  $T_p$  is connected. Now consider a path of Thurston maps  $g_t$ , such that postcritical points of  $P$  stay fixed in  $\varphi_0(\partial\mathcal{K}_p)$  and all postcritical points of  $Q$  move from  $\varphi_\infty(\partial\mathcal{K}_q)$  to  $\varphi_0(\partial\mathcal{K}_p)$  along external rays of  $g$ . This deformation is a kind of two-sided pseudo-isotopy from  $g$  to  $g^\theta$ , since marked points may collapse in different ways on both ends, while each component of  $X$  is invariant under each  $g_t$ . By collapsing all components of  $X$  to points and modification at preimages, equivalent quotient maps are obtained

for all  $g_t$ , in particular for  $g$  and  $g^\theta$ , where postcritical points have been identified already in different ways. So we know that  $\tilde{g}^\theta = \tilde{g} \sim f$  and we may consider  $\tilde{g}^\theta$  as an essential map in the sense of Definition 3.10, with  $\Gamma$  consisting of loops around those trees in  $X$ , which contain at least two postcritical points of  $g^\theta$ . So  $g^\theta$  is essentially equivalent to  $f$ , combinatorially equivalent if  $\Gamma = \emptyset$ , and the same applies to the original precapture  $g_\theta$ . ■

## 7 Conclusion

Suppose  $P$  and  $Q$  are postcritically finite polynomials, not in conjugate limbs of the Mandelbrot set. Then the formal mating  $g = P \sqcup Q$  is combinatorially equivalent or essentially equivalent to a rational map  $f \cong P \amalg Q$ . Consider the following implementations of the Thurston Algorithm for the formal mating, which should converge except for numerical cancellations, unless  $f$  is of type  $(2, 2, 2, 2)$ :

**The medusa algorithm** was developed by Christian Henriksen and others under the guidance of John Hamal Hubbard [7]. Start with a Thurston map having marked points on two circles at specific angles; it is equivalent to the formal mating unless there are Misiurewicz parameters of satellite type — then the arguments from Section 6.2 apply. A medusa is a graph connecting the images of marked points, which corresponds to the equator united with external rays from the equator to these points. Its pullback up to homotopy with rational maps provides a unique choice of preimages. Since this is an implementation of the Thurston Algorithm, the marked points and maps should converge according to Theorem 4.3 or to Section 6.2, unless the orbifold of  $f$  has type  $(2, 2, 2, 2)$ .

However, medusa often seems to be numerically unstable even for simple examples: it begins to converge but after 50–100 iterations it oscillates wildly. It is not known whether this is a bug in the implementation, an unlucky choice of numerical parameters, or an unavoidable feature of this algorithm. I had expected that instability would be related to long ray connections converging to periodic points with a multiplier causing spiraling: then the equator would have to spiral as well and cannot be pruned to a homotopic curve with few long segments. But this idea was not confirmed by experiments; e.g., medusa did converge for  $3/7 \sqcup 3/14$  and  $12/31 \sqcup 19/62$ , which have postcritical ray-equivalence classes of length four, and for  $31/96 \sqcup 1/3$  and  $511/1536 \sqcup 1/3$ , which show significant spiraling. On the other hand, it diverged even in cases without postcritical identifications, e.g.,  $1/14 \sqcup 1/4$  and  $19/60 \sqcup 1/3$ . Note also that for the matings  $5/28 \sqcup 13/28$  and  $7/60 \sqcup 29/60$  of type  $(2, 2, 2, 2)$ , the Thurston pullback accumulates on a four-cycle in configuration space according to [27]; medusa shows this behavior initially but oscillates after a few more iterations.

**Triangulations of the sphere** are used by Laurent Bartholdi in the GAP-package IMG [3, 17]. A Thurston map is represented algebraically as a biset [2], and it is easy to combine maps, as in a formal mating, or to apply a Dehn twist. Then a triangulation is constructed from the biset, which represents an isotopy class

of homeomorphisms. It is pulled back to implement the Thurston Algorithm, with appropriate refinement and pruning.

When marked points get close to each other, the pullback is interrupted in the current version and an obstruction is searched instead, based on the assumption that points will be grouped in the observed way. In the case of formal matings with removable obstructions, this approach might be modified such that either the Thurston pullback is restarted with a component map, or such that the iteration is continued to allow a collapse of marked points according to Theorem 4.3. In the latter case, the pullback might become unstable, when a spiraling of marked points requires an excessive refinement of the triangulation.

**Slow mating** according to Section 5 is much simpler to implement [24]. In the case of postcritical identifications, the path is not required to follow a spiraling equator. So there is a good chance to converge with just a small number of segments per marked point, and the algorithm will still be fast with a large number of segments. However, any discretization of a continuous path as a polygonal path should check, whether the exact pullback to piecewise arcs can be replaced homotopically with piecewise line segments again. This is easy in the case of quadratic polynomials [29], but more involved for quadratic rational maps [28]. Note that there may be a trade-off as well when using many small segments: homotopy violations will happen less often, but detecting them will be numerically less stable.

**An initialization by angles** will be more convenient, but slow mating assumes that the parameters  $p$  and  $q$  are given as floating-point approximations. When angles are given instead, either run the spider algorithm first to determine these parameters, or draw the parameter rays and improve the endpoints with Newton method. Alternatively, the slow mating algorithm can be modified such that the marked points are on two circles initially; the pullback would give the same marked points as medusa, but be more stable.

**A precapture** can be implemented according to Initialization 6.2 as well, using an approximation to dynamic rays in this case.

So this paper suggests to treat removable obstructions by ignoring them, which is simple and fast: trust the slow mating algorithm to converge nevertheless. This route is taken naturally by equipotential gluing as well [11]. If you want to collapse ray-equivalence classes manually, you can determine the relevant angles recursively from the conjugate angle algorithm [26], but the topological part may be harder. When there are only direct connections between postcritical points of  $P$  and  $Q$ , so a pseudo-equator exists [36], the modification can be done by taking a medusa with all points on the equator. Mary Wilkerson [57, 58] has an alternative implementation in this case: the pullback is controlled by a finite subdivision rule, which is constructed from Hubbard trees.

## References

- [1] M. Arfeux, G. Cui, Arbitrary large number of non trivial rescaling limits, preprint (2016). [arXiv:1606.09574](https://arxiv.org/abs/1606.09574)
- [2] L. Bartholdi, V. Nekrashevych, Thurston equivalence of topological polynomials, *Acta Math.* **197**, 1–51 (2006).
- [3] L. Bartholdi, IMG, Computations with iterated monodromy groups, a GAP package, version 0.1.1; [laurentbartholdi.github.io/img/](https://laurentbartholdi.github.io/img/)
- [4] L. Bartholdi, D. Dudko, Algorithmic aspects of branched coverings, preprint (2015). [arXiv:1512.05948](https://arxiv.org/abs/1512.05948)
- [5] B. Bielefeld, Yu. Fisher, J. H. Hubbard, The classification of critically preperiodic polynomials as dynamical systems, *J. Am. Math. Soc.* **5**, 721–762 (1992).
- [6] M. Bonk, D. Meyer, *Expanding Thurston Maps*, [arXiv:1009.3647](https://arxiv.org/abs/1009.3647)
- [7] S. Hruska Boyd, C. Henriksen, The Medusa Algorithm for Polynomial Matings, *Conf. Geom. Dyn.* **16**, 161–183 (2012).  
The code was ported to standard C++ by Chris King, [dhushara.com/DarkHeart/](https://dhushara.com/DarkHeart/)
- [8] H. Bruin, D. Schleicher, Admissibility of kneading sequences and structure of Hubbard trees for quadratic polynomials, *J. Lond. Math. Soc. II. Ser.* **78**, 502–522 (2008).
- [9] X. Buff, Cui G.-Zh., Tan L., Teichmüller spaces and holomorphic dynamics, in: *Handbook of Teichmüller theory IV*, Soc. math. europ. 2014, 717–756.
- [10] A. Chéritat, Tan Lei and Shishikura’s example of non-mateable degree 3 polynomials without a Levy cycle, *Ann. Fac. Sc. Toulouse* **21**, 935–980 (2012).
- [11] A. Chéritat, W. Jung, Slow mating and equipotential gluing, in preparation (2017).
- [12] G. Gui, W. Peng, L. Tan, On a theorem of Rees–Shishikura, *Ann. Fac. Sc. Toulouse* **21**, 981–993 (2012).
- [13] A. Douady, Systèmes dynamiques holomorphes, *Astérisque* **105–106**, 39–63 (1983).
- [14] A. Douady, J. H. Hubbard, A proof of Thurston’s topological characterization of rational functions, *Acta Math.* **171**, 263–297 (1993).
- [15] W. Floyd, W. Parry, K. M. Pilgrim, Presentations of NET maps, preprint (2017). [arXiv:1701.00443](https://arxiv.org/abs/1701.00443)
- [16] R. Funahashi, M. Taniguchi, The Cross-ratio Compactification of the Configuration Space of Ordered Points on  $\widehat{\mathbb{C}}$ , *Acta Math. Sinica, Engl. Ser.* **28**, 2129–2138 (2012).
- [17] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.7; 2017. [www.gap-system.org](http://www.gap-system.org)
- [18] Godillon, A family of rational maps with buried Julia components, *Ergod. Th. Dynam. Sys.* **35**, 1846–1879 (2014).
- [19] P. Haïssinsky, K. Pilgrim, *Coarse expanding conformal dynamics*, *Astérisque* **325**, 2009.

- [20] J. H. Hubbard, D. Schleicher, The Spider Algorithm, in: *Complex Dynamical Systems: Proc. Symp. Appl. Math.* **49**, AMS 1994.
- [21] J. H. Hubbard, S. Koch, An analytic construction of the Deligne–Mumford compactification of the moduli space of curves, *J. Diff. Geom.* **98**, 261–313 (2014).
- [22] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics I: Teichmüller theory*. Matrix editions 2006.
- [23] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics II: Surface Homeomorphisms and Rational Functions*. Matrix editions, 2016.
- [24] W. Jung, Mandel, software available from [www.mndynamics.com](http://www.mndynamics.com)
- [25] W. Jung, Core entropy and biaccessibility of quadratic polynomials I, preprint (2014). [arXiv:1401.4792](https://arxiv.org/abs/1401.4792)
- [26] W. Jung, Quadratic matings and ray connections, in preparation (2017).
- [27] W. Jung, Quadratic matings and Lattès maps, in preparation (2017).
- [28] W. Jung, Quadratic captures and anti-matings, in preparation (2018).
- [29] W. Jung, The Thurston Algorithm for quadratic polynomials, in preparation (2018).
- [30] S. Koch, Teichmüller theory and critically finite endomorphisms, *Adv. Math.* **248**, 573–617 (2013).
- [31] S. Koch, K. Pilgrim, N. Selinger, Pullback invariants of Thurston maps, *Trans. Am. Math. Soc.* **368**, 4621–4655 (2016).
- [32] I. Mashanova, V. Timorin, Captures, Matings and Regluings, *Ann. Fac. Sc. Toulouse* **21**, 877–906 (2012).
- [33] C. Matheus, Lecture notes on the dynamics of the Weil-Petersson flow, preprint (2016). [arXiv:1601.00690](https://arxiv.org/abs/1601.00690)
- [34] C. T. McMullen, *Complex Dynamics and Renormalization*, *Annals of Mathematics Studies*. **135**, Princeton 1995.
- [35] C. T. McMullen, The moduli space of Riemann surfaces is Kähler hyperbolic, *Ann. Math.* **151**, 327–357 (2000).
- [36] D. Meyer, Invariant Peano curves of expanding Thurston maps, preprint (2009). [arXiv:0907.1536](https://arxiv.org/abs/0907.1536)
- [37] J. Milnor, Geometry and dynamics of quadratic rational maps, *Exp. Math.* **2**, 37–83 (1993).
- [38] J. Milnor, Periodic Orbits, External Rays and the Mandelbrot Set: An Expository Account, *Astérisque* **261**, 277–333 (2000).
- [39] J. Milnor, Pasting together Julia sets: a worked out example of mating, *Exp. Math.* **13**, 55–92 (2004).
- [40] J. Milnor, *Dynamics in One Complex Variable*, *Annals of Mathematics Studies*. **160**, Princeton 2006.



- [41] Y. Minsky, Extremal length estimates and product regions in Teichmüller space, *Duke Math. J.* **83**, 249–286 (1996).
- [42] C. L. Petersen, D. Meyer, On the Notions of mating, *Ann. Fac. Sc. Toulouse* **21**, 839–876 (2012).
- [43] K. M. Pilgrim, Canonical Thurston obstructions, *Adv. Math.* **158**, 154–168 (2001).
- [44] K. M. Pilgrim, An algebraic formulation of Thurston’s characterization of rational functions, *Ann. Fac. Sc. Toulouse* **21**, 1033–1068 (2012).
- [45] M. Rees, A partial description of the Parameter Space of Rational Maps of Degree Two: Part 1, *Acta Math.* **168**, 11–87 (1992).
- [46] M. Rees, *The parameter capturemap for  $V_3$* , preprint (2012). [arXiv:1201.4082](https://arxiv.org/abs/1201.4082)
- [47] D. Schleicher, Rational Parameter Rays of the Mandelbrot Set, *Astérisque* **261**, 405–443 (2000).
- [48] N. Selinger, Thurston’s pullback map on the augmented Teichmüller space and applications, *Invent. Math.* **189**, 111–142 (2012).
- [49] N. Selinger, Topological characterization of canonical Thurston obstructions. *J. Mod. Dyn.* **7**, 99–117 (2013).
- [50] N. Selinger, M. Yampolsky, Constructive geometrization of Thurston maps and decidability of Thurston equivalence, *Arnold Math. J.* **1**, 361–402 (2015).
- [51] M. Shishikura, Trees associated with the configuration of Herman rings, *Ergod. Th. Dynam. Sys.* **9**, 543–560 (1989).
- [52] M. Shishikura, On a theorem of Mary Rees, in *The Mandelbrot Set, Theme and Variations*, Tan L. ed., LMS Lecture Notes **274**, Cambridge University Press 2000.
- [53] M. Shishikura, Tan L., A family of cubic rational maps and matings of cubic polynomials, *Exp. Math.* **9**, 29–53 (2000).
- [54] Tan L., Matings of quadratic polynomials, *Ergod. Th. Dyn. Sys.* **12**, 589–620 (1992).
- [55] D. P. Thurston, A positive characterization of rational maps, preprint (2016). [arXiv:1612.04424](https://arxiv.org/abs/1612.04424)
- [56] V. Timorin, Topological regluing of rational functions, *Invent. math.* **179**, 461–506 (2010).
- [57] M. Wilkerson, Subdivision rule constructions on critically preperiodic quadratic matings, preprint (2016). [arXiv:1601.07111](https://arxiv.org/abs/1601.07111)
- [58] M. Wilkerson, Thurston’s algorithm and rational maps from quadratic polynomial matings, preprint (2017). [arXiv:1705.01184](https://arxiv.org/abs/1705.01184)
- [59] B. Wittner, *On the bifurcation loci of rational maps of degree two*, Ph.D. thesis Cornell University 1986.
- [60] S. A. Wolpert, Geometry of the Weil-Petersson completion of Teichmüller space, *Surv. Differ. Geom.* **VIII**, 357–393 (2003).

## A The spider algorithm

In [29], the Thurston Algorithm with a path in moduli space is implemented for quadratic polynomials, including the spider algorithm, twisted polynomials, pre-capture and recapture, and slow tuning. This section sketches the discussion of the spider algorithm, because it is another application of the convergence Theorem 3.11; in fact it was the original motivation for this research.

For an angle  $\theta \in \mathbb{Q} \setminus \mathbb{Z}$ , we want to determine the associated postcritically finite parameter  $c$  of a quadratic polynomial  $f_c(z) = z^2 + c$ . From  $\theta$  a Thurston map  $g_\theta$  is constructed, and the Thurston Algorithm shall give  $f_c$ . Denote the iterates of  $\theta = \theta_1$  by  $\theta_i = 2^{i-1}\theta$ ,  $i \geq 1$ , and the preperiod and period of  $\theta$  is  $k$  and  $p$ . Consider the map  $g_\theta = \varphi_\theta \circ F$  with  $F(z) = z^2$ . Here the homeomorphism  $\varphi_\theta$  is the identity in most of  $\widehat{\mathbb{C}}$ ; it shifts the straight ray with angle  $\theta_1$  radially out by 1, and if  $k = 0$ , it shifts the ray with angle  $\theta_p$  in by 1. So  $\varphi_\theta(0) = e^{i2\pi\theta_1}$ , and in the periodic case  $\varphi_\theta(e^{i2\pi\theta_p}) = 0$ . The straight spider is invariant under  $g_\theta$ .

To apply the Thurston Algorithm, we need to pull back marked points  $x_i(n)$  with quadratic polynomials. The choice of branch for the square roots is determined by the pullback of an isotopy class of homeomorphisms. The basic idea of the **spider algorithm** is: Teichmüller space is represented by spiders, homotopy classes of graphs with legs from  $\infty$  to the marked points, which are pulled back with the polynomials. According to [20, 5, 23] we may consider these cases:

**Case 1:** The angle  $\theta$  is periodic and  $c$  is the center associated to the root  $\gamma_M(\theta)$ . Then  $g_\theta$  is combinatorially equivalent to  $f_c$  and unobstructed. Under the equivalence, the spider legs are homotopic to external rays extended by internal rays, which will have common points in the satellite case.

**Case 2:** The angle  $\theta$  is preperiodic and the Misiurewicz point  $c = \gamma_M(\theta)$  is an endpoint or of primitive type. Again,  $g_\theta$  is unobstructed and equivalent to  $f_c$ . The spider legs correspond to external rays at the postcritical orbit.

**Case 3:** The Misiurewicz point  $c = \gamma_M(\theta)$  is of satellite type; the angle  $\theta_{k+1}$  has period  $p = rq$  and the landing point has period  $q < p$ . Now  $g_\theta$  has a Lévy cycle with  $q$  curves, each containing  $r$  marked points. By identifying these points manually, or by extending the spider legs accordingly, a modified Thurston map  $\tilde{g}_\theta$  is defined; it is unobstructed and combinatorially equivalent to  $f_c$ .

See [20] for a convergence proof in the periodic case, which replaces Teichmüller space with a more explicit spider space. The essential spider map  $\tilde{g}_\theta$  is constructed in [5], and the relation between obstructions, kneading sequences, and the satellite case is obtained in [23]. Note that the description above assumes landing properties of parameter rays according to [47, 38], and the spider algorithm is just a method to compute parameters numerically. Alternatively, one may discuss the spider map  $g_\theta$  directly and conclude the existence of quadratic polynomials with specific combinatorics. There are several variants of implementing the spider algorithm:

- In a pullback step, each leg and endpoint has two preimages under the quadratic polynomial, or the preimage is the critical point with two legs. To

choose unique preimages, either employ the cyclic order of rays at  $\infty$ , which is related to intersection numbers, or consider the angles of the legs at  $\infty$ .

- Either normalize the position of two finite marked points, or assume that all polynomials are of the form  $z^2 + c_n$ . This increases the dimension of Teichmüller space by one and gives an additional eigenvalue  $\lambda = 1/2$ .
- Each leg is encoded as a sequence of points, such that the curve is homotopic to a polygonal curve with respect to the marked points. Since the preimages of straight lines are hyperbolas in general, this means that each hyperbola segment is replaced with a line segment again; we must check that it is homotopic. When this condition is violated in the current step for one or more segments, we may either refine the discretization there (and prune somewhere else), or restart with an overall finer discretization.

In the satellite Misiurewicz case 3, Hubbard–Schleicher [20] observed that colliding marked points converge to postcritical points of  $f_c$  and the polynomials converge to  $f_c$ . To understand this process in general, Selinger [48, 49] considered the extension of the Thurston pullback to augmented Teichmüller space and the dynamics on the canonical stratum. This phenomenon motivated the research for the convergence Theorem 3.11 as well. Intuitively, the points must collide because the unique obstruction is pinched, and since they stay close together while moving, the pullback of  $g_\theta$  shall be similar to the pullback defined by  $\tilde{g}_\theta$  or  $f_c$ . But this description involves interchanging limits, so it is not obvious that the marked points get close to the expected limit and stay there long enough to be attracted.

**Theorem A.1 (following Hubbard–Schleicher and Selinger)**

*For the pullback defined by the unmodified spider map  $g_\theta$ , the polynomials converge to  $f_c$  and the marked points converge to postcritical points, with suitable collisions in the satellite Misiurewicz case 3.*

**Proof:** According to the references given above, either  $g_\theta$  or  $\tilde{g}_\theta$  is unobstructed and equivalent to  $f_c$ . In case 3, the Thurston pullback for  $g_\theta$  diverges due to the Lévy cycle. The essential map  $\tilde{g}_\theta$  is equivalent to  $f_c$  and the other component maps are homeomorphisms. So Theorem 3.11 applies and gives convergence immediately.

Recall the following steps of its proof. In the context of Proposition 3.12 the current situation was called scenario 2: the pullback in configuration space extends to a neighborhood of the prospective limit. The eigenvalues either come from the modified Thurston pullback, or they are of the form  $\lambda^{rq} = \rho^{-r}$ ,  $\lambda^q \neq \rho^{-1}$ , where  $\rho$  is the repelling multiplier of the  $q$ -cycle of  $f_c$ . The techniques of Selinger show that the points in configuration space get arbitrarily close to the prospective limit, such that a segment of an invariant path in Teichmüller space projects into an attracting neighborhood of that configuration. Then it cannot happen that at some step another branch of the pullback relation becomes active, so the points do not jump away. ■

In contrast to the situation of formal matings, this generalized convergence property is not crucial from a numerical perspective, since the modification from  $g_\theta$  to  $\tilde{g}_\theta$  is simple and explicit. As a completely different approach, the parameter  $c$  may be obtained by drawing the parameter ray  $\mathcal{R}_M(\theta)$  and starting a Newton iteration from

the approximate endpoint. Now, let us consider an alternative implementation of the spider algorithm, which pulls back a path in moduli space instead of spiders in Teichmüller space. So the legs are invisible, but the movement of the feet is recorded:

**Initialization A.2 (Spider algorithm with a path)**

Suppose  $\theta = \theta_1 \in \mathbb{Q} \setminus \mathbb{Z}$  has preperiod  $k$  and period  $p$ . Define  $(x_1(t), \dots, x_{k+p}(t))$  for  $0 \leq t \leq 1$  as

$$\begin{aligned} x_1(t) &= t \cdot e^{i2\pi\theta_1} \\ x_p(t) &= (1-t) \cdot e^{i2\pi\theta_p}, \quad \text{if } k=0 \\ x_j(t) &= e^{i2\pi\theta_j}, \quad \text{otherwise.} \end{aligned} \tag{25}$$

Pull this path back continuously with  $x_i(t+1) = \pm\sqrt{x_{i+1}(t) - x_1(t)}$ . Then it converges to the marked points of  $f_c$  with appropriate collisions.

**Proof:** The argument is similar to that given for captures and precaptures according to Initialization 6.2, and for twisted maps according to Examples 3.1 and 3.7 in [27]. We may initialize the Thurston pullback for  $g_\theta = \varphi_\theta \circ F$  by  $\psi_0 = \varphi_\theta^{-1}$  and  $\psi_1 = \text{Id}$ . There is an obvious deformation  $\psi_t$  along one or two rays, which projects to the path defined in moduli space. By Proposition 2.4, this shows that the pullback of the path agrees with the projection of the pullback in Teichmüller space. Note that for  $k=0$ , we have  $x_p(t) = 0$  only for  $t \geq 1$ . Likewise, for  $k=1$  the relation  $x_{k+p}(t) = -x_k(t)$  is satisfied for  $t \geq 1$  only. ■

This algorithm gives the same marked points as the spider algorithm with legs, and it converges unless there are floating-point cancellations or problems with the discretization: again, the path is represented by a polygonal curve, and there is an explicit check for homotopy violations by the simultaneous deformation of hyperbola segments to line segments; if that happens, refine or restart. Since only a path of length  $|n \leq t \leq n+1| = 1$  needs to be stored instead of full legs, we may take a large number of line segments easily, but there is a trade-off: there will be little need for refinement, because small hyperbola segments are close to small line segments, but there is a loss of precision by subtracting numbers that are approximately equal.

For exponential functions with preperiodic singular value, spiders and modified spiders are constructed in [SZ, LSV], and convergence of unobstructed pullback maps follows from [HSS]. The alternative implementation with a path in moduli space is straightforward, but a check for homotopy violations will be harder. Examples show convergence of colliding marked points analogously to Theorem A.1. While the local analysis at the prospective limit is the same, the extension to augmented Teichmüller space is unknown and so the global analysis is incomplete.

[HSS] J. Hubbard, D. Schleicher, M. Shishikura, Exponential Thurston maps and limits of quadratic differentials, *J. AMS* **22**, 77–117 (2009).

[LSV] B. Laubner, D. Schleicher, V. Vicol, A Combinatorial Classification of Postsingularly Finite Complex Exponential Maps, *Discrete cont. dyn. systems* **22**, 2008.

[SZ] D. Schleicher, J. Zimmer, Periodic points and dynamic rays of exponential maps, *Ann. Acad. Scient. Fenn.* **28**, 327–354 (2003).