

# Quadratic matings and Lattès maps

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## Abstract

In complex dynamics the Thurston Theorem decides, if a postcritically finite branched covering  $g$  is equivalent to a rational map  $f$ . In the general case this happens, if and only if there is no obstructing multicurve. In the exceptional case of orbifold type  $(2, 2, 2, 2)$ , however, the criterion is different. A Thurston map of this kind is described by a real-affine map on a torus, and there is an equivalent Lattès map, if and only if the  $2 \times 2$  matrix has non-real eigenvalues, or it is a multiple of the identity. Note that when  $g$  of type  $(2, 2, 2, 2)$  is unobstructed, there need not be an equivalent rational map  $f$ .

A geometric mating of two postcritically finite quadratic polynomials  $P(z) = z^2 + p$  and  $Q(z) = z^2 + q$  is a rational map  $f$ , which is conjugate to the topological mating, where the filled Julia sets of  $P$  and  $Q$  are glued along their boundaries. Moreover,  $f$  is combinatorially equivalent to the formal mating  $g$  or to the essential mating  $\tilde{g}$ . When there are suitable identifications between postcritical points,  $f$  may be a Lattès map. Using polynomial combinatorics, it is shown here that there are precisely nine kinds of examples, such that  $\tilde{g}$  is of type  $(2, 2, 2, 2)$ , and the parameters  $p$  and  $q$  are not in conjugate limbs of the Mandelbrot set. It turns out that a rational map  $f$  exists in every case. In the general situation of a hyperbolic orbifold, the corresponding result was obtained by Rees–Shishikura–Tan [TL], who showed that  $\tilde{g}$  is unobstructed. In the exceptional case of type  $(2, 2, 2, 2)$ , we need to check the eigenvalues of the matrix associated to  $\tilde{g}$ , which is done by applying the Shishikura Algorithm to each example individually.

The direct or indirect identifications between postcritical points are understood by collapsing ray-equivalence classes of the formal mating  $g$ , which defines the essential mating  $\tilde{g}$ . Since these identifications correspond to removable obstructions of  $g$ , the Thurston pullback for  $g$  diverges to the boundary of Teichmüller space. According to [quadJung], the sequence or path of rational maps does converge to  $f$  nevertheless, when the orbifold of  $\tilde{g}$  is not of type  $(2, 2, 2, 2)$ : the postcritical ray-equivalence classes are pinched automatically. Now  $g$  has hyperbolic orbifold but  $\tilde{g}$  is of type  $(2, 2, 2, 2)$ . In a joint work with Chéritat, it is shown that for one of the nine kinds of matings, the slow mating algorithm converges to  $f$ . In eight cases, the path spirals towards a one-dimensional center manifold instead.

# 1 Introduction

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A Lattès map of type  $(2, 2, 2, 2)$  is a postcritically finite rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  with four postcritical points, whose critical points are non-degenerate and not postcritical. It is projected from an unbranched covering of a torus, or from an affine map  $\mathbb{C} \rightarrow \mathbb{C}$  modulo  $w \mapsto -w$ . Thurston maps of type  $(2, 2, 2, 2)$  are described by affine maps of  $\mathbb{R}^2$ , and many properties are obtained explicitly from integer matrices.

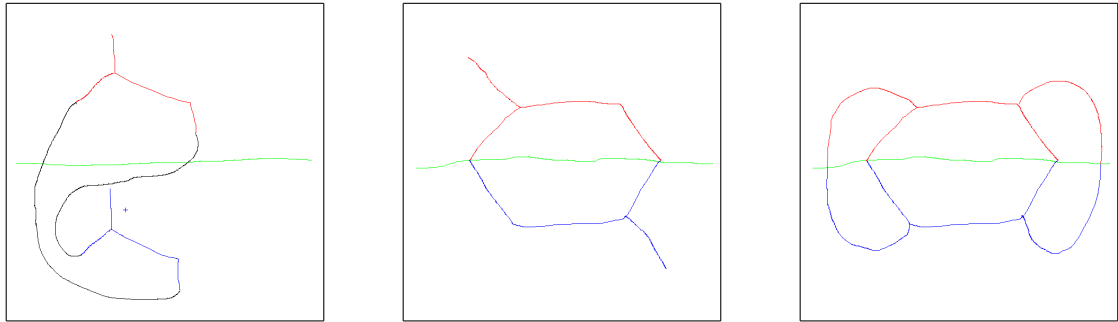
iteration, pcf, Thurston

remainder in paragraphs corresponding to sections

dfns mating, question of convergence for  $(2, 2, 2, 2)$  orbifold

refs Milnor 1/4 v 1/4 several aspects, including gamma and tilings and measure, here answering question on uniqueness of a semi-conjugation

For the Lattès map of type  $(2, 4, 4)$ , there are precisely three kinds of matings, and the slow mating algorithm converges in each case.



Fig

**Figure 1:** The formal self-mating  $1/6 \sqcup 1/6$  is shown in the left cartoon by drawing the Hubbard trees in blue and red, the equator in green, and two postcritical ray connections in black. These are simple rays with the angles  $1/3$  and  $2/3$ , respectively. The middle sketch illustrates the essential mating, where these ray connections are collapsed. The preimage on the right shows that these structures are not invariant under pullback, but there are more branches and more identifications.

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## 2 Lattès maps and Thurston maps

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### 2.1 Lattès maps

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Consider a lattice  $\Lambda = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \xi \subset \mathbb{C}$  with  $\text{Im}(\xi) > 0$ , and an affine map  $L(w) = \eta \cdot w + \kappa$  with  $\eta \cdot \Lambda \subset \Lambda$ , which covers a self-map of the torus  $\mathbb{C}/\Lambda$ . If  $\Lambda$  and  $L$  have additional symmetries, we obtain a postcritically finite rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  of degree  $d = |\eta|^2$ , a Lattès map. Under the symmetry  $w \mapsto -w$  in particular, with  $\kappa \in \Lambda/2$ ,  $f$  has four postcritical points and all critical points are non-degenerate

and not postcritical, which is symbolized by the orbifold type  $(2, 2, 2, 2)$ . There is an even Weierstraß function  $\wp : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  with  $f \circ \wp = \wp \circ L$ . Probably the best known examples are flexible or integral Lattès maps, which exist when  $d$  is a square: for  $\eta = \sqrt{d}$  the generator  $\xi$  is arbitrary, giving a one-parameter family of quasi-conformally conjugate rational maps, which have invariant line fields. We shall see that already in degree  $d = 2$ , Lattès maps have many interesting properties, e.g., related to the Thurston theory and to matings. See [pasteMilnor, lattesMilnor, BM] for other properties and alternative characterizations.

Figure like [SY] or [pasteMilnor]. explain geometry  $\Lambda, \Lambda/2$ , torus, pillowcase, sphere. covering is branched on  $\Lambda/2$ , maps to postcritical set  $P$  of  $f$ .  $f^{-1}(P) = \Omega \cup P$

From now on, we assume the degree is  $|\eta|^2 = 2$ . Since  $\eta \cdot \Lambda \subset \Lambda$ , there are integers  $a, b, c, d$  such that

$$\eta \cdot 1 = a + c\xi \quad \eta \cdot \xi = b + d\xi . \tag{1}$$

eqxyabcd

Due to the symmetry  $w \mapsto -w$ ,  $f$  is covered by another map as well, where  $\eta$  is replaced with  $-\eta$ , but  $\eta^2$  characterizes  $f$ . Different choices of  $\kappa$  may give equivalent maps, cf. Proposition 2.1.2. A change of  $\xi$  means choosing a different fundamental cell in the same lattice  $\Lambda$ , and the matrix  $A$  with components  $a, b, c, d$  will be conjugated with a matrix  $S \in SL_2(\mathbb{Z})$ . In the quadratic rational case, we always have  $bc \neq 0$ , and  $\text{Im}(\eta) = \text{Im}(c\xi) \neq 0$ . Now (1) gives the following relations,

$$\eta^2 - (a + d)\eta + 2 = 0 \quad c\xi^2 + (a - d)\xi - b = 0 , \tag{2}$$

eqxyquad

and the determinant is  $ad - bc = 2$ . In particular, we have  $|a + d| \leq 2$  and there are only finitely many values possible for  $\eta^2$ . Moreover, there are only three branch portraits of type  $(2, 2, 2, 2)$  in degree two. Grouping complex conjugate maps together, it turns out there are four cases of quadratic rational maps of type  $(2, 2, 2, 2)$ ; an overview is given in Table 1.

## 2.2 The Thurston characterization

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define Thurston map  $g$ , postcritical set  $P$ , marked set  $Z$ , Thurston's combinatorial equivalence  $\sim$

question of rational map, reference to obstructions

application to constructing rational maps, and matings in particular.

A quadratic Thurston map of type  $(2, 2, 2, 2)$  can be constructed as follows: take the lattice  $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$ , fix an even cover  $\mathbb{R}^2/\mathbb{Z}^2 \rightarrow \widehat{\mathbb{C}}$ , choose an affine map  $L(\vec{x}) = A\vec{x} + \vec{\kappa}$  with determinant 2 and  $\vec{\kappa} \in \mathbb{Z}^2/2$ , and define  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that it is covered by  $L$ . The notation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad : \quad \begin{pmatrix} d \\ -c \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} -b \\ a \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{3}$$

eqabcdxy

is compatible with (1) and (2); so  $\eta$  is an eigenvalue of  $A$  and  $\xi$  corresponds to the second base vector, if it is not real. Conversely, every quadratic Thurston map  $g$  of type  $(2, 2, 2, 2)$  is equivalent to a map covered by an affine map in this way. See [DH, book2, BM, SY] for the proof, which is based on the intermediate lift to a torus and on the identification of a homology group with  $\mathbb{Z}^2$ , or on the notion of a universal orbifold cover.

Quadratic rational Lattès maps $f$ of type $(2, 2, 2, 2)$	
Branch portrait a) and b) $\Rightarrow \bullet \searrow$ $\bullet \rightarrow \bullet \uparrow$ $\Rightarrow \bullet \nearrow$	c) $\Rightarrow \bullet \rightarrow \bullet$ $\downarrow$ $\Rightarrow \bullet \rightarrow \bullet$ d) $\Rightarrow \bullet \rightarrow \bullet \uparrow$ $\Rightarrow \bullet \rightarrow \bullet \uparrow$
a) $c = \pm i, \eta^2 = \mp 2i, \kappa = 0$ b) $u = 1 \pm \sqrt{2}, \eta^2 = -2, \kappa = 0$	c) $c = \frac{1 \pm \sqrt{7}i}{4}, \eta^2 = \frac{-3 \mp \sqrt{7}i}{2}, \kappa = 1/2$ d) $c = \frac{1 \pm \sqrt{7}i}{2}, \eta^2 = \frac{-3 \pm \sqrt{7}i}{2}, \kappa = 0$
a) is symmetric under inversion, b) under complex conjugation.	Both c) and d) are symmetric under inversion.
The Thurston pullback $\sigma_f$ , the pullback of simple closed curves, and the virtual endomorphism $\Phi_f : H \rightarrow G$ of the pure mapping class group are of finite order.	All of these relations have no periodic orbits except for the obvious fixed points.
In a symmetric cover, $\sigma_f$ projects to an isomorphism of moduli space, and a moduli space map $k$ exists there.	The correspondence on moduli space is not reduced in a covering space.
Some iterate of $f$ is a flexible Lattès map.	No iterate of $f$ is flexible.
Quadratic Thurston maps $g$ of Lattès type $(2, 2, 2, 2)$	
The trace of $A$ is even.	The trace is odd.
All values of $\vec{\kappa}$ are equivalent.	Changing $\vec{\kappa}$ gives either c) or d).
$g$ cannot have an obstruction.	$g$ may be obstructed.

Ucomp

**Table 1:** Up to conjugation with a Möbius transformation or complex conjugation, there are four cases of quadratic rational maps of type  $(2, 2, 2, 2)$ ; see Sections 2.4 and 2.5 for concrete formulas. Thurston maps of type  $(2, 2, 2, 2)$  are discussed in Sections 2.2 and 2.3.

**Proposition 2.1** ()

Pconj1

1. A quadratic Thurston map  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is of type  $(2, 2, 2, 2)$ , if it has four postcritical points and no critical point is postcritical. Assuming that there are no additional marked points, it is combinatorially equivalent to a map covered by a real affine map  $L(\vec{x}) = A\vec{x} + \vec{\kappa}$  on the torus  $\mathbb{R}^2/\mathbb{Z}^2$  modulo  $\vec{x} \mapsto -\vec{x}$ .

2. The possible branch portraits according to Table 1 are related to the parity of the trace  $t = a + d$ :

- When  $t$  is even, the branch portrait is a)b), and all values of the translation  $\vec{\kappa} \in \mathbb{Z}^2/2$  give conjugate affine maps  $L$ .
- When  $t$  is odd, changing  $\vec{\kappa}$  gives two different affine conjugacy classes, which correspond to the branch portraits of case c) and d), respectively.

3. Suppose the Thurston maps  $g$  and  $\tilde{g}$  are covered by the affine maps  $L(\vec{x}) = A\vec{x} + \vec{\kappa}$  and  $\tilde{L}(\vec{x}) = \tilde{A}\vec{x} + \tilde{\kappa}$ , respectively. Then  $\tilde{g}$  is combinatorially equivalent to  $g$ , if and only if  $\tilde{A}$  is conjugate to  $\pm A$  with a conjugator  $S \in SL_2(\mathbb{Z})$ , and the translations are conjugate as well.

**Proof:** 1. See the references given above, and [SY] for the case of additional marked points; then a lift is possible unless there are removable Lévy cycles.

2. The branch portrait is obtained by checking all combinations of parity; a

priori there are sixteen combinations, but a few are ruled out by determinant 2. An explicit calculation checks whether two translations in  $\mathbb{Z}^2/2$  are equivalent in  $\mathbb{Z}^2$  by conjugating with a translation in  $\mathbb{Z}^2/2$ .

3. Combinatorial equivalences  $\psi_0, \psi_1$  are covered by maps isotopic to the same affine map; this map sends  $\mathbb{Z}^2/2$  to itself, so  $S \in SL_2(\mathbb{Z})$ . We cannot distinguish between  $A$  and  $-A$ , since the covering identifies  $\vec{x}$  and  $-\vec{x}$ . ■

**Theorem 2.2 (Thurston characterization)**

TTp1

Consider a quadratic Thurston map  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  of type  $(2, 2, 2, 2)$ , which is covered by the real affine map  $L(\vec{x}) = A\vec{x} + \vec{\kappa}$  up to isotopy. Then  $g$  is combinatorially equivalent to a rational map  $f$ , if and only if the eigenvalues of  $A$  are not real, or equivalently, if the trace is  $|t| \leq 2$ .

**Proof:** If the eigenvalues are not real, determine  $\xi$  and  $\eta$  from (1) and (2) with  $\text{Im}(\xi) > 0$ . Conjugate  $L$  with an affine map  $\mathbb{R}^2 \rightarrow \mathbb{C}$ , which is sending the base vectors to 1 and  $\xi$ . The new map will be of the form  $\eta \cdot w + \kappa$ , so it is covering a rational Lattès map  $f$ . Conversely, any rational  $f$  is described by an affine map with non-real eigenvalues according to the previous Section 2.1, and the matrices must be conjugate by Proposition 2.1.3. ■

**Remark 2.3 (Lifting curves)**

Rpresh

In practice, the affine lift of a Thurston map  $g$  can be obtained as follows: choose a simple closed curve  $\gamma$  through the four postcritical points, and cover  $\widehat{\mathbb{C}}$  by  $\mathbb{R}^2/\mathbb{Z}^2$  such that the interior of  $\gamma$  is covered by  $[0, 1/2]^2$ . Lift  $\gamma' = g^{-1}(\gamma)$  to  $\mathbb{R}^2$ . The curves will be  $\mathbb{Z}^2$ -periodic and isotopic to straight lines through lattice points. Choose a fundamental cell and determine the affine map  $L$  sending this parallelogram to  $[0, 1]^2$ . The coefficients of  $A$  are read off from (3). Of four adjacent cells, two will give an orientation-preserving map, and these two give  $\pm A$ .

Equivalence  $S^{-1}AS = \pm \tilde{A}$  with a conjugator  $S \in SL_2(\mathbb{Z})$  can be checked algorithmically. It is related to a classical problem in number theory in fact, the classification of integral quadratic forms. By work of Lagrange, Legendre, Gauß, and Zagier, e.g., there are effective reduction algorithms. — We shall use the normal forms

$$C_t^+ = \begin{pmatrix} 0 & -2 \\ 1 & t \end{pmatrix}, \quad C_t^- = \begin{pmatrix} 0 & 2 \\ -1 & t \end{pmatrix}. \tag{4}$$

eqcomp

$C_t^+$  is the companion matrix of trace  $t = a + d$  and the second matrix is chosen such that  $A$  is conjugate to  $C_t^-$ , if  $-A$  is conjugate to  $C_{-t}^+$ . So an idea to determine equivalence classes for some modulus  $|t|$  of the trace would be to check whether every matrix is conjugate to  $C_t^\pm$ , and whether these two are conjugate. Both questions lead to representations by quadratic forms: are there  $x, y \in \mathbb{Z}$  with

$$cx^2 + (d - a)xy - by^2 = \pm 1 \quad \text{or} \quad -x^2 + txy - 2y^2 = 1, \tag{5}$$

eqxyform

respectively? In both equations, the discriminant is  $\Delta = t^2 - 8$ . In simple cases, we may either find an explicit solution or see a contradiction, avoiding the general reduction theory.

### Example 2.4 (Rational classes, jointly with Michael Mertens)

Xcmmmm2

When  $|t| \leq 2$ , then  $g$  is equivalent to a rational map. For each value of  $t$ , there are two conjugacy classes of matrices, and there are two classes up to sign with  $|t| = 1$  or  $|t| = 2$ , respectively. The case of  $t = 0$  is special:  $C_0^+$  and  $C_0^-$  are not conjugate, but  $C_0^- = -C_0^+$ , so there is only one combinatorial equivalence class. Now  $t = 2$  can be treated with the unique factorization of Gaussian integers  $\mathbb{Z}[i]$  as well: When  $A$  has  $c > 0$ , there are  $k, m, n \in \mathbb{Z}$  with  $k, m > 0$  and  $km = n^2 + 1$ , such that

$$A = \begin{pmatrix} 1-n & -m \\ k & 1+n \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (6)$$

eqtrtwo

The ansatz  $AS = S\tilde{A}$  to determine  $S$  gives two equations, which are equivalent to a single complex equation  $k(\alpha + i\beta) = -(n+i)(\gamma + i\delta)$ . Now the prime factorization of  $km = (n+i)(n-i)$  gives  $u, v \in \mathbb{Z}[i]$  with  $n+i = uv$ ,  $k = |u|^2$ , and  $m = |v|^2$ . Then  $S$  is obtained as  $\alpha + i\beta = v$  and  $\gamma + i\delta = -\bar{u}$ , which has determinant  $\text{Im}(uv) = 1$ . — In a way, this argument can be reverted, so that the Thurston Theorem implies that every divisor of  $n^2 + 1$  is a sum of two squares.

### Example 2.5 (Non-rational classes, jointly with Michael Mertens)

Xcmmmm3

When  $|t| \geq 3$ ,  $g$  is not equivalent to a rational map. For  $A$  of trace  $t = 3$ , we use  $-bc = (a-1)(a-2)$  to show that  $A$  is diagonalized with  $S \in SL_2(\mathbb{Z})$ . For  $4 \leq t \leq 11$ , the class number varies between one and two; these can be determined online from  $\Delta = t^2 - 8$ . See [www.numbertheory.org/php/classnpos0.html](http://www.numbertheory.org/php/classnpos0.html). For  $t = 12$  there are four equivalence classes of quadratic forms, or conjugacy classes of matrices, and four combinatorial equivalence classes:

$$\begin{pmatrix} 0 & \mp 2 \\ \pm 1 & 12 \end{pmatrix} \quad \begin{pmatrix} 2 & \pm 6 \\ \pm 3 & 10 \end{pmatrix} \quad (7)$$

eqclassnum

While these class numbers count primitive quadratic forms, so no prime number divides all three coefficients, it may happen in our case that  $(d-a)$  shares a common factor with  $b$  and  $c$ . This requires a prime  $p$  with  $p^2 | (t^2 - 8)$ , and  $p = 2$  is excluded by determinant 2. The first example is obtained for  $t = 20$  and  $p = 7$ : there are only two classes of primitive quadratic forms with  $\Delta = 392$ , but at least four combinatorial equivalence classes. Checking (5) for the second matrix in (8) gives  $\pm 7x^2 + 14xy \mp 7y^2 = 1$  or  $= -1$ , so it is not conjugate to a companion matrix.

$$\begin{pmatrix} 0 & \mp 2 \\ \pm 1 & 20 \end{pmatrix} \quad \begin{pmatrix} 3 & \pm 7 \\ \pm 7 & 17 \end{pmatrix} \quad (8)$$

eqclassmult

## 2.3 The Thurston pullback

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dfn Teichmüller space, moduli space, and pullback map  $\sigma_g$ ; fixed point gives equivalence to rational map [DH, book2, teich]

In the previous Section 2.2, we have characterized Thurston maps of type  $(2, 2, 2, 2)$  in terms of the trace  $t = a + d$  of an associated matrix  $A$ , and it was not necessary to consider the pullback map. Now  $\sigma_g$  shall be discussed as well for two reasons: to compare the pullback behavior to maps of type not  $(2, 2, 2, 2)$ , see also Remark ???. And because the convergence properties of the Thurston Algorithm for certain matings will be investigated in Section 4; it turns out that locally in the

higher-dimensional space, there is an invariant center manifold, where the pullback behaves as in the case of type (2, 2, 2, 2).

**Theorem 2.6 (Thurston pullback)**

TTp2

Suppose  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is a quadratic Thurston map of type (2, 2, 2, 2), without additional marked points. Then Teichmüller space  $\mathcal{T}$  is identified with the upper half-plane, and the Thurston pullback is given by a Möbius transformation

$$\sigma_g(\tau) = \frac{d\tau + b}{c\tau + a} \tag{9}$$

eqsigmah

with integer coefficients and determinant 2. There are two possible cases:

- $\sigma_g$  has a unique fixed point in the upper halfplane  $\mathcal{T}$ . Then  $g$  is combinatorially equivalent to a rational Lattès map  $f$ , which is unique up to Möbius conjugation. The fixed point is neutral with multiplier  $\rho = 2/\eta^2$ .
- $\sigma_g$  does not have a fixed point in  $\mathcal{T}$ , there is no equivalent rational map, and the Thurston pullback  $\tau_n = \sigma_g^n(\tau_0)$  diverges to the boundary of  $\mathcal{T}$ .

**Proof:** explain pullback of constant Beltrami coefficient This gives (9), where  $a, b, c, d$  are the coefficients of the matrix  $A$ . Now  $\sigma_g(\tau) = \tau$  yields the same equation as (2) for  $\xi$ , so a fixed point  $\tau$  in the upper halfplane exists, if and only if there is a  $\xi$  with  $\text{Im}(\xi) > 0$ . A short computation gives  $\sigma'_g(\xi) = 2/\eta^2$ . ■

**Remark 2.7 (Thurston obstructions)**

Rlattesobstr

When  $g$  is not of type (2, 2, 2, 2), the Thurston pullback is weakly contracting, so a fixed point is unique and globally attracting. An obstructing multicurve  $\Gamma$  for a Thurston map  $g$  has certain properties under pullback, which imply that in the Riemann surfaces defined by  $\tau_n$ , corresponding hyperbolic geodesics get shorter and annuli get thicker, which may prevent convergence. When  $g$  is not of type (2, 2, 2, 2), the Thurston Theorem says that  $g$  is equivalent to a rational map, if and only if it is unobstructed. This is not true if  $g$  is of Lattès type (2, 2, 2, 2):

- When  $g$  is quadratic and the matrix  $A$  from the affine lift of  $g$  has trace  $|t| \geq 4$ , there will be no obstruction, but  $g$  is not equivalent to a rational map either. The pullback is bounded in moduli space, but diverges to the boundary in Teichmüller space.
- Only for  $|t| = 3$ , there is an obstruction, and the iteration diverges to the boundary in moduli space as well, since the obstruction is pinching. According to Selinger [char], invariant essential curves are related to integer eigenvectors of  $A$ , and so determinant 2 requires trace  $\pm 3$ . Note that an obstructed map will be of case c) or d) according to Proposition 2.1.2. Alternatively, the core arc argument shows that maps of case a)b) are unobstructed, since an invariant essential curve would contradict the branch portrait.
- Quadratic rational maps are always unobstructed, but when the degree is a square, there exists a family of flexible Lattès maps, which have a non-pinching obstruction in fact.

**2.4 The rational maps of cases a) and b)**

2e

Now we shall determine the quadratic rational Lattès maps  $f$  of type (2, 2, 2, 2) explicitly, by observing the branch portraits visualized in Table 1. See also [pasteMil-

nor, lattesMilnor]. The pullback correspondence on moduli space is discussed as well. The affine maps from the previous Sections 2.2–2.3 will not be used to obtain  $f$ ; but to see which rational map corresponds to which value of  $\eta$ , either the relation  $\rho = 2/\eta^2$  can be employed, or the fact that a fixed point of  $f$  has multiplier  $\pm\eta$  if it is not postcritical, and multiplier  $\eta^2$  if it is.

The branch portrait of cases a) and b) is the same: both critical values are mapped to the same prefixed point. Assume that the critical points are 0 and  $\infty$ , so that  $f$  is even, and put the postcritical fixed point at 1, so  $f(\pm 1) = 1$ . Denoting the critical values by  $f(0) = u$  and  $f(\infty) = -u$ , we have functions of the form  $F_u$  with

$$F_u(z) = -u \frac{z^2 + \frac{1+u}{1-u}}{z^2 - \frac{1+u}{1-u}} \quad F_u^{-1}(z) = \sqrt{\frac{1+u}{1-u} \cdot \frac{z-u}{z+u}}. \quad (10)$$

eqffinbab

These functions will be used both for the Thurston pullback, where the parameter  $u$  varies, and to determine specific values of  $u$  representing Lattès maps. Now the condition  $F_u(\pm u) = -1$  gives  $(u^2 + 1)(u^2 - 2u - 1) = 0$ .

**Case a)** shall denote the maps with  $u = \pm i$ ,  $F_{\pm i}(z) = \mp i \frac{z^2 \pm i}{z^2 \mp i}$ . The two maps are complex conjugate to each other, and each is invariant under conjugation with the inversion  $z \mapsto 1/z$ , so it can be written in the form  $f_{\pm i}(z) = \frac{z^2 \pm i}{1 \pm iz^2}$  according to (12) as well. Computing  $F'_u(1) = (1 - u^2)/u$  gives  $\eta^2 = -2i$  for  $f_i$  and  $\eta^2 = 2i$  for  $f_{-i}$ , and we may assume  $\kappa = 0$ .

**Case b)** is given by  $u = 1 \pm \sqrt{2}$ , so  $F_{1 \pm \sqrt{2}}(z) = -(1 \pm \sqrt{2}) \frac{z^2 - (1 \pm \sqrt{2})}{z^2 + (1 \pm \sqrt{2})}$ . The two maps are real in this normalization, not symmetric under inversion, but they are transformed into each other by the inversion in fact; so they belong to the same combinatorial equivalence class. We have  $\eta^2 = F'_{1 \pm \sqrt{2}}(1) = -2$  and  $\kappa = 0$ .

In this normalization at 0 and  $\infty$ , moduli space is given by  $u \in \widehat{\mathbb{C}} \setminus \{0, \infty, -1, 1\}$ . In case a) or b), the Thurston pullback  $\sigma_f$  of  $f$  defines a correspondence on moduli space, such that  $u$  is pulled back to  $u'$ . Now  $u$  determines  $F_u$  by its critical values, and  $u'$  satisfies  $F_u(\pm u') = -1$  since  $F_u(-1) = 1$ . This example has the special property, that the correspondence is reducible: (10) gives

$$u' = \pm \frac{1+u}{1-u}, \quad \mathbf{a)} \quad u' = + \frac{1+u}{1-u} \quad \mathbf{b)} \quad u' = - \frac{1+u}{1-u}. \quad (11)$$

eqsigmaaab

Here the sign is determined from the known values of  $u$  at the fixed point of  $\sigma_f$ ; it is the same sign globally by analytic continuation. The multiplier  $\rho$  of the Thurston pullback is computed either from (11) or from the general relation  $\rho = 2/\eta^2$ , which gives  $\rho = \pm i$  for  $F_{\pm i} = f_{\pm i}$ ,  $\eta^2 = \mp 2i$  of case a), and  $\rho = -1$  in case b).

### Remark 2.8 ()

R2e

1. The Thurston pullback map is of finite order,  $\sigma_f^4$  or  $\sigma_f^2$  is the identity. This can be seen either from the fact that it is a Möbius transformation of the upper halfplane with a rationally neutral fixed point, or by noting that at its fixed point,  $\sigma_f$  is analytically conjugate to a branch of the correspondence (11) on moduli space, and employing analytic continuation. Note that  $\pi : \mathcal{T} \rightarrow \mathcal{M}$  is an infinite-to-one cover semiconjugating the Möbius transformation  $\sigma_f$  to the Möbius transformation (11). When  $g$  is a Thurston map of type (2, 2, 2) with branch portrait a)b), the correspondence on moduli space is given by (11) as well, but  $\sigma_f$  will not be of finite order, if it does not have a fixed point.



2. In case b),  $f^2$  is a flexible Lattès map of degree four. Since  $f$  is represented as a geometric mating according to Theorem 3.3, the composition is represented by a mating, whose combinatorial equivalence class does not determine a unique Möbius conjugacy class; this observation is due to Pilgrim [pasteMilnor]. — Note that  $\sigma_f$  can be used to obtain a parametrization for the flexible family: with  $u' = -(1+u)/(1-u)$ , define  $I_u = F_u \circ F_{u'}$ . Analogously, case a) could be used to discuss the flexible Lattès family of degree sixteen.

3. The correspondence on moduli space is reducible only in the “even” normalization, which is a covering of ordinary moduli space in fact, see also Theorem ?? and Example ??.

4. Since the maps  $f_{\pm i}$  are symmetric under inversion, we may look at the pullback map restricted to symmetric maps. It turns out that this map is actually constant, since  $f_c(\pm i) = -1$  for all parameters  $c$ . The multiplier  $\rho$  of  $\sigma_f$  should not depend on the normalization, but this use of symmetric maps is not a normalization. Locally there are two invariant manifolds, one with even maps and multiplier  $\mp i$ , one with symmetric maps and multiplier 0; the first step of the pullback lands on the even one, and symmetric maps land on the fixed point. We shall see in Section 4.2 that slow mating converges for the self-mating of  $z^2 + \gamma_M(1/4)$ , which is related to the eigenvalue of the invariant manifold being 0 instead of neutral.

## 2.5 The rational maps of cases c) and d)

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Consider the following one-parameter families of quadratic rational maps, with  $c \neq \pm 1$ ,  $u \neq 0, 1$ , or  $u \neq \pm 1$ . Again they are normalized with critical points 0 and  $\infty$ , and  $f_c$  is symmetric with respect to conjugation by the inversion  $z \mapsto 1/z$ :

$$f_c(z) = \frac{z^2 + c}{1 + cz^2} \quad H_u(z) = \frac{z^2 - \frac{u+1}{2}}{z^2 - \frac{u+1}{2u}} \quad h_u(z) = \frac{z^2 - \frac{2u}{u+1}}{z^2 - \frac{2}{u+1}} \quad (12)$$

eqfHh

**Case c)** denotes a Lattès map with a postcritical 2-cycle. It shall have the following branch portrait:  $0 \Rightarrow u \rightarrow -1 \leftrightarrow -u \leftarrow 1 \Leftarrow \infty$ . This is provided by  $H_u$  if  $u$  satisfies  $H_u(\pm u) = -1$ , or  $(u-1)(4u^2 + 3u + 1) = 0$ . So there are two complex conjugate maps  $H_u$  with  $u = (-3 \pm \sqrt{7}i)/8$ . It turns out they are rescaled to symmetric maps  $f_c$  with  $c = (1 \pm \sqrt{7}i)/4$ . Then  $\pm \eta = f'_c(1)$  shows  $\eta^2 = (-3 \mp \sqrt{7}i)/2$  for the affine lift, and  $\kappa = 1/2$  gives the correct branch portrait.

Now consider the Thurston pullback with  $\pi(\tau) = u$  and  $\pi(\sigma_f(\tau)) = u'$  in the even normalization  $H_u$ . The correspondence on moduli space is determined from  $H_u(\pm u') = -1$  as

$$H_u^{-1}(z) = \sqrt{\frac{u+1}{2u} \cdot \frac{z-u}{z-1}}, \quad u' = H_u^{-1}(-1) = \frac{u+1}{2\sqrt{u}}. \quad (13)$$

eqpbrcu

At the parameters  $u = (-3 \pm \sqrt{7}i)/8$ , a branch has a neutral fixed point with the multiplier  $\rho = (-3 \pm \sqrt{7}i)/4 = 2/\eta^2$ . Note that  $\rho^2 + \frac{3}{2}\rho + 1 = 0$  shows that the fixed point is irrationally neutral; I do not know whether it is Brjuno, but a local branch of (13) will be linearizable anyway, because it is conjugate to the Möbius transformation  $\sigma_f$ . The pullback relation has a superattracting fixed point  $u = 1$  in addition, which does not correspond to a rational map, but indicates that Thurston maps with branch portrait c) may have a pinching obstruction.

**Case d)** is a Lattès map with disjoint critical orbits, such that both critical values are mapped to fixed points. Now  $h_u$  according to (12) satisfies  $\infty \Rightarrow 1 \rightarrow -1 \uparrow$ , and  $0 \Rightarrow u \rightarrow -u \uparrow$  requires  $h_u(\pm u) = -u$  or  $u(u-1)(u^2+3u+4) = 0$ ; here  $u = 1$  is excluded and  $u = 0$  has a different branch portrait. So  $u = (-3 \pm \sqrt{7}i)/2$  gives two complex conjugate Lattès maps  $h_u$ . Again, these are rescaled to symmetric maps  $f_c$  with  $c = (1 \pm \sqrt{7}i)/2$ . We have  $\eta^2 = (-3 \pm \sqrt{7}i)/2$  and  $\kappa = 0$ .

The Thurston pullback induces a correspondence on moduli space;  $h_u(\pm u') = -u$  gives

$$h_u^{-1}(z) = \sqrt{\frac{2}{u+1} \cdot \frac{z-u}{z-1}}, \quad u' = h_u^{-1}(-u) = \frac{2\sqrt{u}}{u+1}. \quad (14)$$

eqpbrdu

The irrationally neutral fixed points at  $u = (-3 \pm \sqrt{7}i)/2$  have the multipliers  $\rho = (-3 \mp \sqrt{7}i)/4$ . Again, there is a superattracting fixed point at  $u = 1$  indicating a possible pinching obstruction;  $u = 0$  is not attracting.

In both cases c) and d) the Teichmüller space and moduli space contain another invariant manifold corresponding to symmetric maps. The pullback relation reads

$$\text{c) } c' = \sqrt{-\frac{c^2+1}{2c}}, \quad \text{d) } c' = \sqrt{-\frac{2c}{c^2+1}}. \quad (15)$$

eqpbrcdc

These pullback relations are locally conjugate to (13) and (14), respectively, via  $u = c^2$ . So they have the same neutral multiplier  $\rho$  at corresponding fixed points, in contrast to case a) according to Remark 2.8.4. — Note that the affine lifts of cases c) and d) have the same  $\eta^2$  but differ in the translation  $\kappa$ ; the rational maps are related, e.g., as follows: if  $f_c$  is of case c), then  $f_{1/c}$  is of case d), and  $f_c^2 = f_{1/c}^2$ .

### 3 Lattès maps as matings

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#### 3.1 Polynomial dynamics and combinatorics

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P, K, rays, landing, persistence behind root

In Sections 3.3 and 5, we shall need special results on periodic cycles, to find or to exclude certain types of ray connections, and to characterize essential matings with specific ramification portraits. Item 1 is proved by counting endpoints of Hubbard trees, and items 2 and 3 mean that a rotation number with high denominator is rigid with respect to small changes. See also Proposition 3.6.c in [raysJung].

#### Lemma 3.1 (Combinatorics of quadratic polynomials)

L1m

Consider a Misiurewicz polynomial  $P(z) = z^2 + p$ .

1. Suppose  $p$  has preperiod  $k$ , and the corresponding periodic cycle of  $P^k(p)$  persists from the root  $p' \prec p$ . Then  $k \geq 2$ , and  $k = 2$  occurs only when  $p$  is real.

2. Suppose  $p$  has preperiod  $k$  and it belongs to a limb of denominator  $r$ . The periodic cycle of  $P^k(p)$  shall have the same angles as the  $\alpha$ -fixed point of another limb of the same denominator  $r$ :

a) If  $k = 1$ , then  $r = 3$ .

b) If  $k = 2$  and the two limbs are conjugate, then  $r = 3$  or  $r = 4$ .

3. Suppose  $p, \tilde{p}$  are Misiurewicz points of preperiod 1 and belong to limbs of denominators  $r, \tilde{r}$ . Now  $P(p)$  shall have an angle of  $\alpha_{\tilde{r}}$  and  $\tilde{P}(\tilde{p})$  shall have an angle of  $\alpha_p$ . If  $\tilde{r} < r$ , then  $\tilde{r} = 2$  and  $r = 3$ .

Recall that for each hyperbolic component with root  $p' \neq 1/4$ , there is an associated cycle of primitive or satellite type, whose rays persist for all parameters  $p \succeq p'$ . Conversely, if a periodic cycle of  $P$  does not consist of endpoints, there will be a corresponding root  $p' \preceq p$ . In particular, when  $p$  belongs to the limb of rotation number  $s/r$ , the fixed point  $\alpha_p$  has an  $r$ -cycle of dynamic rays.

**Proof:** 1. First, assume that  $p'$  is the root of a limb with rotation number  $s/r$ . Then  $P^k(p) = \alpha_p$  requires  $k \geq r$ , so  $k = 1$  is excluded, and  $k = 2$  only for  $r = 2$  and the real parameter  $p = \gamma_M(5/12) = \gamma_M(7/12)$ . Second, assume the periodic cycle is  $z_1, \dots, z_m$  with  $m \geq 3$  and the characteristic point  $z_1$  separating  $p$  from the other points in the cycle.  $T$  is the Hubbard tree of  $P$  and  $T' \subset T$  the connected hull of the  $m$ -cycle. Then  $z_1$  and  $z_2$  are endpoints of  $T'$ , and 0 is an inner point.  $p$  is behind  $z_1$  and  $P(p)$  behind  $z_2$ , i.e.,  $z_2$  is separating  $P(p)$  from 0. If  $k = 1$ , then  $P(p)$  is a periodic point  $z_j$  behind  $z_2$ , which contradicts  $z_2$  being an endpoint of  $T'$ . If  $k = 2$ , then  $z_3$  is not an endpoint of  $T'$ , because  $P^2(p)$  would be a periodic point behind it, noting that  $P$  is injective on  $[z_1, p]$  and on  $[z_2, P(p)]$ , and an arc before  $z_2$  would be mapped before  $z_3$ . So  $T'$  has only two endpoints, and  $P^2(p) \in T'$  implies that  $T$  has two endpoints as well, so  $p$  is real.

2a) For  $r = 2$ , there is no other limb of the same denominator. For  $r = 3$ ,  $p = \gamma_M(\pm 3/14)$  belongs to the limb with rotation number  $\pm 1/3$ , and it is mapped to the angle  $\pm 3/7$ , which belongs to  $\alpha$  of the conjugate limb. For  $r \geq 4$  we shall obtain a contradiction: Denote the sectors at  $\alpha_p$  by  $W_1, \dots, W_r$  in the order of the orbit of  $p$ , with  $p \in W_1$  and  $0 \in W_r$ . The periodic  $r$ -cycle of  $P(p)$  shall be labeled such that it has corresponding indices, so  $z_2 = P(p) \in W_2, \dots, z_r = P^{r-1}(p) \in W_r$ , and  $z_1 = P^r(p)$ . Now  $z_1$  is the periodic preimage of  $z_2$ , so  $z_1 = -p$  is behind  $-\alpha_p$  and belongs to  $W_r$ . The periodic points are endpoints by item 1, and we are interested in the cyclic order of their angles  $\theta_j$ . Since  $\theta_r$  and  $\theta_1$  are the only angles in  $W_r$ , the rotation number must be  $\pm 1/r$ . Compare these angles to the original sectors: we have removed position 1 and added a new position 1 next to position  $r$ . If  $r \geq 4$ , there are at least two neighboring positions left unchanged, so the rotation number was  $\pm 1$  in the limb of  $p$  already. This contradicts the hypothesis and item 1.

2b) For  $r = 3$  or  $r = 4$  we have  $p = \gamma_M(\pm 5/28)$  and  $p = \gamma_M(\pm 7/60)$ , respectively. For  $r = 5$  and  $r = 6$ , no solution is found. It remains to obtain a contradiction for  $r \geq 7$ : We have  $p \in W_1, P(p) \in W_2, P^2(p) = z_3 \in W_3, \dots, P^{r-1}(p) = z_r \in W_r, P^r(p) = z_1$ , and  $P^{r+1}(p) = z_2 = -P(p) \in W_r$ . Now  $z_1$  is mapped into  $W_r$ , so  $z_1 \in W_{r-1}$  or  $z_1 \in W_r$ .

Case 1:  $z_1 \in W_{r-1}$  and in  $W_r$  we have, say, the cyclic order  $z_r$  before  $z_2$ . Then the new rotation number is  $s'/r$  with  $s' = (r+1)/2$ , so the old one was  $s/r$  with  $s = (r-1)/2$ . It turns out that compared to the order of the original sectors, two neighboring positions are swapped two times; position 1 is swapped with  $r-1$ , and position  $r$  swapped with 2. But there are other positions jumping over two neighbors, so the rotation number could not have changed.

Case 2:  $z_1 \in W_r$  and in  $W_r$  we have  $s'$  steps from  $z_r$  to  $z_1$  and from there to  $z_2$  as well. So without restriction assume  $s' = 1$ . Then  $z_{r-1}$  comes directly before  $z_r$  regarding the cyclic order of angles. Since we have no periodic points in  $W_1$  and

$W_2$ , only these could be between  $W_{r-1}$  and  $W_r$ , so the old number of steps  $s$  was 1, 2, or 3. This contradicts  $s + s' = r$ .

3. For  $r = 3$  and  $\tilde{r} = 2$ , we have  $p = \gamma_M(\pm 1/6)$  and  $\tilde{p} = \gamma_M(\mp 5/14)$ . So suppose  $p$  has rotation number  $s/r$  with  $r \geq 4$ . We may assume  $s/r < 1/2$ . Denoting the sectors at  $\alpha_p$  by  $W_1, \dots, W_r$  again, the periodic points are in  $W_2, \dots, W_r$ : the latter sector is the first one mapped back to  $W_2$ , so  $\tilde{r} = r - 1$ . There are  $2s - 1$  steps from  $z_r$  to its image  $z_2$  and  $s$  steps from  $z_{r-1}$  to  $z_r$ . So  $s = 1$ ,  $\alpha_p$  has rotation number  $1/r$  and  $\alpha_{\tilde{p}}$  has  $1/(r - 1)$ . The possible angles of the endpoint  $\tilde{p}$  are determined from the inequality  $\frac{1}{2^{r-1}-1} < \frac{5}{2(2^r-1)} < \frac{7}{2(2^r-1)} < \frac{2}{2^{r-1}-1}$ . However, doubling the two angles in the middle does not give an angle of  $\alpha_p$ , which is of the form  $\frac{2^j}{2^r-1}$ . ■

## 3.2 Definitions of mating

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### Theorem 3.2 ()

Trstrs

*pcf non-conjugate limbs: RST, Th, RS*

## 3.3 Lattès maps of type (2, 2, 2, 2) as matings

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Up to inversion and complex conjugation, we have four rational maps  $f$  and nine matings  $g$  to consider. Shishikura has found seven of these matings and determined, which formal mating  $g$  corresponds to which rational function  $f$ . His algorithm is described in [pasteMilnor] and in Section 3.4. The results are reported in the following Table 2. Interchanging  $P$  and  $Q$  conjugates the mating with an inversion, and reflection of both angles means complex conjugation of  $P$  and  $Q$  and of the rational map. Altogether we have thirty matings for eight rational maps up to affine conjugation, or seven rational maps up to Möbius conjugation.

	$L(w) = \eta w + \kappa$	$\rho =$	$f_c, F_u$	mating	anti-mating
a)	$\kappa = 0, \eta^2 = 2i$	$-i$	$c = -i$	$f \cong 3/4 \amalg 3/4$ $f \simeq 5/28 \amalg 13/28$ $f \simeq 7/60 \amalg 29/60$	$f \cong 1/4 \amalg 1/4$
b)	$\kappa = 0, \eta^2 = -2$	$-1$	$u = 1 \pm \sqrt{2}$	$f \simeq 1/12 \amalg 5/12$	—
c)	$\kappa = 1/2,$ $\eta^2 = \frac{-3 + \sqrt{7}i}{2}$	$\frac{-3 - \sqrt{7}i}{4}$	$c =$ $\frac{1 - \sqrt{7}i}{4}$	$f \cong 5/6 \amalg 5/6$	$f \cong 3/14 \amalg 3/14$
d)	$\kappa = 0,$ $\eta^2 = \frac{-3 + \sqrt{7}i}{2}$	$\frac{-3 - \sqrt{7}i}{4}$	$c =$ $\frac{1 + \sqrt{7}i}{2}$	$f \cong 1/6 \amalg 5/14$ $f \cong 3/14 \amalg 3/14$ $f \simeq 3/14 \amalg 1/2$ $f \simeq 5/6 \amalg 1/2$	$f \cong 5/6 \amalg 5/6$

Ulist

**Table 2:** According to Definition 2.2 in [raysJung],  $\cong$  means the rational map is topologically conjugate to the topological mating in the usual normalization, angle 0 at  $z = 1$ , and  $\simeq$  indicates that a rotation of the fixed points is applied in addition. The symmetric anti-matings are obtained from the following result [antiJung]: if  $f_c \cong P \amalg P$ , then  $f_{1/c} \cong P \amalg P$ . There may be further representations by non-symmetric anti-matings.

**Theorem 3.3 (Lattès matings, following Shishikura)**

Tlm

1. *There are precisely 30 formal matings  $g = P \sqcup Q$  of quadratic polynomials, such that the essential mating  $\tilde{g}$  has a parabolic orbifold of type  $(2, 2, 2, 2)$ , and the parameters  $p$  and  $q$  are not in conjugate limbs of the Mandelbrot set. Up to complex conjugation and interchanging  $P$  and  $Q$ , these matings are represented by the nine matings in Table 2.*

2. *In each case, the essential mating  $\tilde{g}$  is combinatorially equivalent to a rational map  $f \simeq P \amalg Q$ , which is given in the table as well. So  $f$  is a geometric mating in fact, conjugate to the topological mating.*

The nine kinds of formal matings are obtained below, and the corresponding rational maps are identified as combinatorial matings in Section 3.4 from the Shishikura Algorithm; this completes the proof of the Rees–Shishikura–Tan Theorem 3.2 for orbifold type  $(2, 2, 2, 2)$ . By the Rees–Shishikura Theorem [RS], the combinatorial mating is a geometric mating as well. To prove there are only nine cases up to Möbius transformation and complex conjugation, we shall employ the following ideas:

- If  $\tilde{g}$  has a postcritical fixed point, a postcritical point of  $g$  must belong to a fixed ray-equivalence class. By an observation of Sharland [...], a ray-equivalence class fixed by  $g$  must contain a fixed point of  $P$  or  $Q$ . See Proposition 2.6 in [raysJung] for a more detailed description of rational ray-equivalence classes.
- The ray-equivalence class of  $\beta$  is a single ray, but the class of  $\alpha$  provides more possibilities. To build longer ray connections, rays from different cycles are joined at periodic points, which persist from primitive hyperbolic components before the current parameters. In principle these connections can be arbitrarily long, but when a ray-equivalence class contains an  $\alpha$ -fixed point, there will be no primitive hyperbolic component of the same ray period in that limb.
- For a Misiurewicz point of low preperiod  $k$  in a limb of high ray period  $r$ , the corresponding periodic cycle will follow the rotation for several steps; in certain situations, this places a restriction on  $r$ . Specific results were obtained in Lemma 3.1 from polynomial combinatorics.

We shall frequently speak of rays with angle  $\theta$  connecting  $\mathcal{K}_p$  and  $\mathcal{K}_{\bar{q}}$ ; this gives an accurate description of the combinatorics without taking complex conjugate angles all the time, but geometrically it means that the  $\theta$ -ray of  $\mathcal{K}_p$  is joined with the ray of angle  $-\theta$  at  $\mathcal{K}_{\bar{q}}$ .

**Proof of uniqueness for the branch portrait of cases a)b):** Since the essential mating maps both critical values to the same prefixed point,  $P^2(p)$  and  $\bar{Q}^2(\bar{q})$  must belong to the same ray-equivalence class, which is fixed by the formal mating  $g$ . If this is the 0-ray, we have  $p = q = \gamma_M(\pm 1/4)$  since  $p = \bar{q}$  is excluded. Otherwise this class contains an  $\alpha$ -fixed point of  $P$  or  $\bar{Q}$ ; by Möbius conjugation we may assume it to be  $\alpha_p$ , as the branch portrait is symmetric.

1. Suppose  $P^2(p) = \alpha_p$ , then  $p$  is real by Lemma 3.1.1 since the preperiod is  $k = 2$ . So  $p = \gamma_M(5/12) = \gamma_M(7/12)$  and the only remaining angles of the same denominator are  $1/12$  and  $11/12$ . Taking one of these for  $\bar{q}$  is seen to work, since  $\bar{q}$  is not in the same limb as  $p$ , and  $\bar{Q}^2(\bar{q})$  shares an angle with  $\alpha_p$ .

2. Now suppose that  $P^2(p)$  is connected to  $\alpha_p$ . This connection goes through only

one primitive cycle of  $\bar{Q}$  and  $P^2(p)$  is an endpoint of the ray-equivalence class, since there is only one hyperbolic component with the ray period of  $\alpha_p$  in the limb of  $p$ . Thus  $\bar{Q}^2(\bar{q})$  must belong to the same primitive cycle of  $\bar{Q}$ , and by Lemma 3.1.1 again,  $\bar{q}$  is real. So the cycle is real and joins complex conjugate angles; the angle of  $P^2(p)$  is complex conjugate to an angle of  $\alpha_p$ . Now Lemma 3.1.2b says that  $p$  is in a limb of ray period 3 or 4. Each of these limbs has a unique angle with the required denominator, which defines  $p$ , and a unique  $\bar{q}$  is found to work: it is real and the primitive cycle at  $\bar{Q}^2(\bar{q})$  shares angles with both  $P^2(p)$  and  $\alpha_p$ . This gives  $\pm 5/28 \amalg 13/28$  and  $\pm 7/60 \amalg 29/60$ . ■

**Proof of uniqueness in case c):** In the essential mating, both critical values shall be mapped to the unique 2-cycle. For  $\pm 1/6 \amalg \pm 1/6$  this works, because the 2-cycles of  $P$  and  $\bar{Q}$  have direct ray connections. See Figure 1. Suppose we had a different formal mating with a 2-cycle of ray-equivalence classes, which contain  $P(p)$  and  $\bar{Q}(\bar{q})$ , respectively. Without restriction, the 2-cycle of  $P$  is of satellite type and forms the symmetry centers of these ray-equivalence classes, and  $p$  is in a sublimb of the period-2 component of  $\mathcal{M}$ . Since the preperiod is  $k = 1$ , Lemma 3.1.1 shows that  $p$  is an endpoint of  $\mathcal{M}$  and  $P(p)$  is an endpoint of its ray-equivalence class. So  $\bar{Q}(\bar{q})$  is an interior point of the other ray-equivalence class; it is of primitive type in contradiction to preperiod  $k = 1$ . ■

**Proof of uniqueness in case d):** In the essential mating  $\tilde{g}$ , the critical values are mapped to different fixed points; in the formal mating  $g$ ,  $P(p)$  and  $\bar{Q}(\bar{q})$  belong to distinct fixed ray-equivalence classes. Up to Möbius conjugation, we have the following possibilities:

1. If both classes contain  $\beta$ -fixed points, so  $p = \bar{q} = -2$ , we are in conjugate limbs. The classes are not actually distinct and the essential mating is undefined, since the critical values would coincide. The topological mating would be defined on a line segment instead of a sphere. In this case, the formal mating is of type  $(2, 2, 2, 2)$  in fact, and it is obstructed with trace  $\pm 3$ .
2. Suppose  $\bar{Q}(\bar{q}) = \beta_{\bar{q}}$  and  $P(p)$  is in the ray-equivalence class of  $\alpha_p$ . Preperiod  $k = 1$  and Lemma 3.1.1 give  $P(p) \neq \alpha_p$ . The ray connection from  $P(p)$  to  $\alpha_p$  passes through a single periodic point of  $\bar{Q}$ , and the angle is complex conjugated since  $\bar{q} = -2$  is real. So  $P(p)$  shares its angle with  $\alpha_{\bar{p}}$ . By Lemma 3.1.2a, we have  $p = \gamma_M(\pm 3/14) \in \mathcal{M}_{\pm 1/3}$ . The angle  $\pm 3/7$  of  $P(p)$  is reflected at the Airplane characteristic point in  $\mathcal{K}_{\bar{q}}$  to become  $\pm 4/7$ , which is an external angle of  $\alpha_p$ .
3. Suppose  $\bar{Q}(\bar{q}) = \beta_{\bar{q}}$  and  $P(p)$  is in the ray-equivalence class of  $\alpha_{\bar{q}}$ , which consists of the rays with angles  $\pm 1/3$ . This gives  $p = \gamma_M(\pm 1/6)$ .
4. Suppose  $P(p)$  is connected to  $\alpha_p$  and  $\bar{Q}(\bar{q})$  is connected to  $\alpha_{\bar{q}}$ . Since  $k = 1$ , Lemma 3.1.1 gives  $P(p) \neq \alpha_p$  and  $\bar{Q}(\bar{q}) \neq \alpha_{\bar{q}}$ . So the former ray-equivalence class contains a primitive cycle of  $\bar{Q}$ , whose period is greater than the ray period of  $\alpha_{\bar{q}}$  and the same as the ray period of  $\alpha_p$ . But by the same arguments, the ray period of  $\alpha_{\bar{q}}$  is greater than that  $\alpha_p$ , which is a contradiction.
5. Suppose  $P(p)$  is connected to  $\alpha_{\bar{q}}$  and  $\bar{Q}(\bar{q})$  is connected to  $\alpha_p$ . These connections must be direct, since a longer connection would require a primitive hyperbolic component before  $p$  but with period exceeding the ray period of that limb, or analogously for the limb of  $\bar{q}$ . So  $P(p)$  shares its angle with  $\alpha_{\bar{q}}$  and  $\bar{Q}(\bar{q})$  shares its angle with  $\alpha_p$ . The ray periods may be equal or different. In the former case, Lemma 3.1.2a gives  $p = q = \gamma_M(\pm 3/14)$ ;  $P(p)$  has the angle  $\pm 3/7$ , which is found at  $\alpha_{\bar{q}}$  as well.

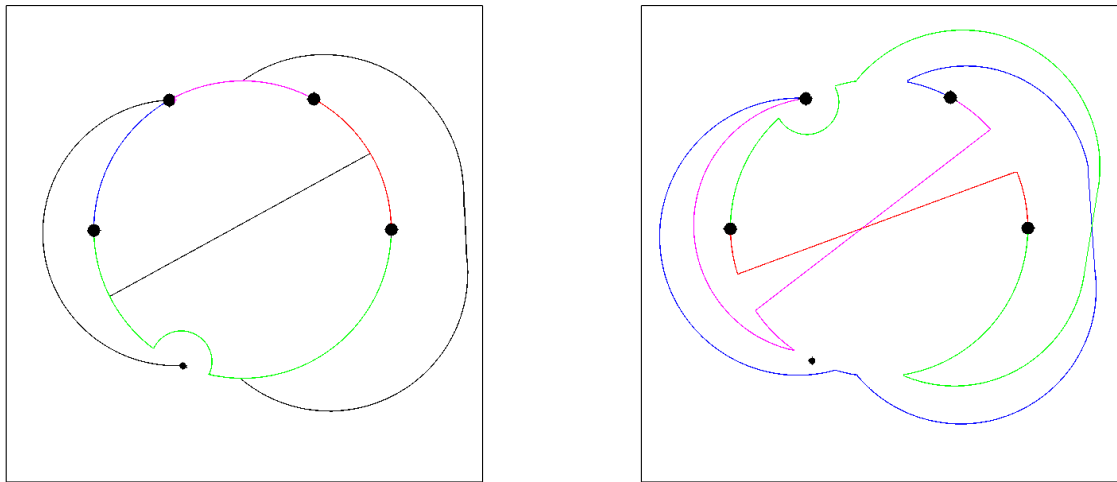
When the ray periods are different, Lemma 3.1.3 gives  $p \in \mathcal{M}_{\pm 1/3}$  and  $\bar{q} \in \mathcal{M}_{1/2}$  or vice versa. So  $P(p)$  has the angle  $1/3$  or  $2/3$ , yielding  $p = \gamma_M(\pm 1/6)$ , and  $\bar{Q}(\bar{q})$  has  $\pm 1/7$ ,  $\pm 2/7$ , or  $\pm 4/7$ . This gives  $\bar{q} = \gamma_M(\pm 9/14)$  and  $q = \gamma_M(\pm 5/14)$ . ■

### 3.4 The Shishikura Algorithm

44

*It is best carried out with a set of colored markers and a quantity of blank paper.*  
 — John Milnor [pasteMilnor]

both proof of matability and identification.  
 note only one shared precapture, which gives that case b) is real.



F1612 Figure 2: ...

**Remark 3.4 (Petersen transformation)**

Rclp

...

## 4 Convergence or divergence of slow mating

— *Joint work with Arnaud Chéritat*

5

### 4.1 Thurston Algorithm and slow mating

51

This is Theorem 4.3 in [quadJung]:

**Theorem 4.1 (Convergence of maps & rational ray-equivalence classes)**

Tsmrcc

...

### 4.2 Convergence in the case of $1/4 \sqcup 1/4$

52

example of case a): 2 parameters, invariant manifold, attracting eigenvalues because of symmetry

**Proposition 4.2 ()**

P2222conv

For the formal mating  $g = 1/4 \sqcup 1/4$ , consider ... slow with any symmetric initialization, eqgl with any radius ...

**Proof:**

■

**Remark 4.3 ()**

R2222conv

1. prf alternatively with trafo or locally
2. not for precapture or other two matings
3. The mating  $1/4 \coprod 1/4$  is discussed in detail by Milnor [pasteMilnor]. He constructs the semi-conjugation from the angle-doubling map by using a linear functional equation on the lattice, and obtains fractal tilings as images of suitable intervals. These results do not rely on the Thurston algorithm. Tilings for Lattès maps are discussed in [lattesCFP] as well. They show that in case d), no self-similar tiling with one initial tile, nor with two Jordan tiles, does exist. An alternative tiling is constructed by Wilkerson [...] using Hubbard trees for the initial subdivision, to describe the essential mating.

**4.3 Divergence in the case of  $1/6 \coprod 1/6$**

53

**4.4 Divergence in all cases except  $1/4 \coprod 1/4$**

54

**5 Lattès maps of type (2, 4, 4)**

6

The notion of an orbifold, its type and its universal cover, is explained in [bookMilnor]. Most types of Thurston maps  $g$  or postcritically finite rational maps  $f$  have hyperbolic orbifold, but there are a finite number of types with parabolic orbifold [DH, book2H]. These maps are covered by affine maps on a cylinder or a torus; the latter are Lattès maps [pasteM, lattesM, BM]. We have three examples of the former in the quadratic case:  $z^{\pm 2}$  is of type  $(\infty, \infty)$  and  $z^2 - 2$  of type  $(2, 2, \infty)$ . The polynomials are trivial matings  $z^2 \cong z^2 \coprod z^2$  and  $(z^2 - 2) \simeq (z^2 - 2) \coprod z^2$ . When only postcritical points are marked, the Thurston pullback of the formal mating is undefined for  $z^2$ , constant for  $z^2 - 2$ . Normalizing the fixed point on the equator to 1, the Thurston Algorithm for  $z^2$  is constant, while for  $z^2 - 2$  it is a rescaling.

Lattès maps of type  $(2, 2, 2, 2)$  have been discussed in the previous sections. There is only one further type in the quadratic case: the branched covering  $\mathbb{C} \rightarrow \widehat{\mathbb{C}}$  or  $\mathbb{C}/\Lambda \rightarrow \widehat{\mathbb{C}}$  and the map  $L$  are symmetric with respect to a quarter rotation, and triangular domains correspond to half-spheres. The rational map  $f$  has three postcritical points, including a critical point that is the image of the other critical point, and the orbifold type is  $(2, 4, 4)$ . Maps of this type are Möbius conjugate to  $f(z) = 1 - 2/z^2$  with the ramification portrait  $0 \Rightarrow \infty \Rightarrow 1 \rightarrow -1 \uparrow$ . As for type  $(2, 2, 2, 2)$ , this map does not occur as a formal mating, but as an essential and geometric mating in cases where the formal mating has hyperbolic orbifold:

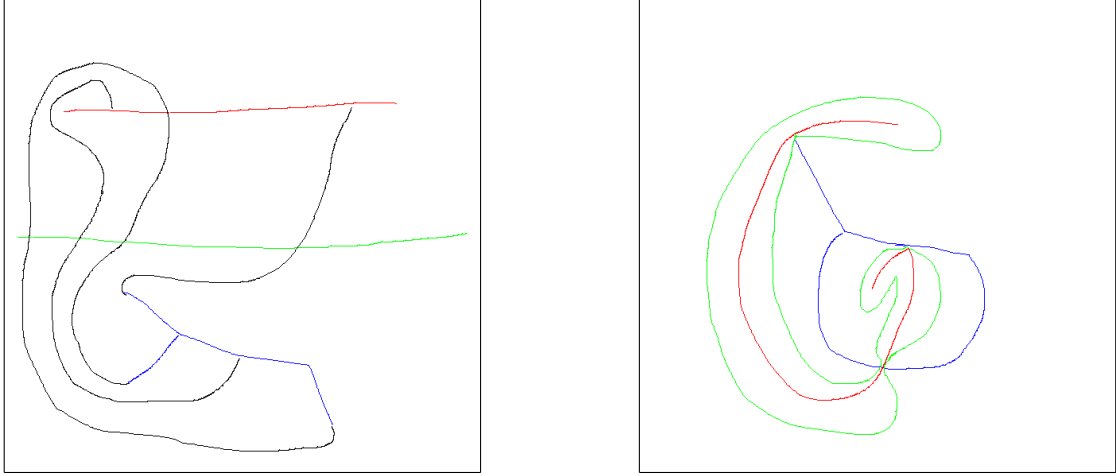
**Theorem 5.1 (Matings and convergence for essential type (2, 4, 4))**

T244

1. The rational map  $f(z) = 1 - 2/z^2$  of type  $(2, 4, 4)$  is a geometric mating with  $f \simeq 1/4 \coprod 1/2 \simeq 5/12 \coprod 1/6 \simeq 13/28 \coprod 3/14$ . These are the only representations up to complex conjugation.
2. In each case, the rational maps from the Thurston Algorithm for the formal mating, or the slow mating algorithm, do converge.



Moreover,  $f$  is given by the geometric anti-mating  $z^2 \amalg (z^2 + q)$  with  $q^3 = -2$ , and the formal anti-mating converges  $f_n \rightarrow f$  as well [antiJung]. Ray connections of  $5/12 \sqcup 1/6$  are illustrated in Figure 3, and the canonical stratum of  $1/4 \sqcup 1/2$  is shown in Figure 2 of [quadJung].



F16512

**Figure 3:** The formal mating  $1/6 \sqcup 5/12$  is shown in the left cartoon by drawing the Hubbard trees in blue and red, the equator in green, and some critical and postcritical ray connections in black, which are subsets of ray-equivalence classes. The right sketch illustrates the essential mating, where some ray connections are collapsed. These structures are not invariant under pullback, but there would be more branches and more identifications. The self-identification of the green curve suggests that there is no simple pseudo-equator for this mating. (The sketch is inspired by Wilkerson [...].)

**Identification of the matings of type  $(2, 4, 4)$ :** We shall use similar arguments as in Section 3.3 and the same notation, labeling critical values of the formal mating  $g = P \sqcup Q$  as  $p$  and  $\bar{q}$ . Once the essential mating  $\tilde{g}$  is shown to have the same ramification portrait as  $f$ , they will be combinatorially equivalent in fact, since there are only three postcritical points and there is only one rational Möbius conjugacy class. Then the geometric and topological matings are obtained from the Rees–Shishikura Theorem [RS]. — So we must determine all  $P$  and  $Q$ , such that  $P^2(p)$  and  $\bar{Q}(\bar{q})$  belong to the same fixed ray-equivalence class:

1. If this class consists of the 0-ray, we have  $\bar{q} = \gamma_M(1/2) = -2 = q$  and thus  $p = \gamma_M(\pm 1/4) \in \mathcal{M}_{\pm 1/3}$ .
2. Suppose this class contains  $\alpha_p$ . If  $P^2(p) = \alpha_p$ ,  $p$  must be real according to Lemma 3.1.1, since the preperiod is  $k = 2$ . So  $p = \gamma_M(5/12) = \gamma_M(7/12)$ , and  $\bar{q} = \gamma_M(\pm 1/6)$  has the property that  $\bar{Q}(\bar{q})$  shares the angle  $\pm 1/3$  with  $\alpha_p$ . Now suppose that there was another example with a longer ray connection from  $P^2(p)$  to  $\alpha_p$ . This connection must have length two, since there is no primitive hyperbolic component of the same ray period in the limb of  $p$ . So there is a unique primitive component before  $\bar{q}$ , such that the cycle persisting behind it shares angles with both  $P^2(p)$  and  $\alpha_p$ . Since there are no other points of  $\mathcal{K}_{\bar{q}}$  in the ray-equivalence class of  $\alpha_p$ , this primitive cycle must contain  $\bar{Q}(\bar{q})$  as well. But this contradicts Lemma 3.1.1 since the preperiod is  $k = 1$ .
3. Suppose the fixed postcritical class contains  $\alpha_{\bar{q}}$ , then  $\bar{Q}(\bar{q})$  is an endpoint connected to  $\alpha_{\bar{q}}$  with length two: the points cannot coincide because the preperiod is

$k = 1$ , and there can be no primitive component of the required period before  $\bar{q}$ , which would give a longer ray connection. So  $P^2(p)$  belongs to the primitive cycle sharing angles with  $\bar{Q}(\bar{q})$  and  $\alpha_{\bar{q}}$ , and preperiod  $k = 2$  implies that  $p$  is real according to Lemma 3.1.1. Now the angle of  $\bar{Q}(\bar{q})$  is complex conjugate to an angle of  $\alpha_{\bar{q}}$  and belongs to  $\alpha_q$  in the conjugate limb. By Lemma 3.1.2a, the ray period is 3. With  $\bar{q} = \gamma_M(\pm 3/14) \in \mathcal{M}_{\pm 1/3}$ , the angle  $\pm 3/7$  of  $\bar{Q}(\bar{q})$  is connected to  $\pm 4/7$  of  $\alpha_{\bar{q}}$  at the cycle persisting from the Airplane; the parameter  $p = \gamma_M(13/28) = \gamma_M(15/28)$  is the only one of preperiod 2 and period 3 behind the Airplane. ■

**Convergence of slow mating for type (2, 4, 4):**  $\infty$  is postcritical and we may mark 0 as well without increasing the dimension. Convergence of the marked points and maps for the Thurston Algorithm of the unmodified formal mating  $g$  is obtained directly from Theorem 4.1, since the orbifold of the essential mating  $\tilde{g}$  is not of type  $(2, 2, 2, 2)$ . But let us look at possible more direct arguments. Three postcritical ray-equivalence classes will be pinched: the critical class of  $p \sim \bar{0}$ , the pre-fixed class of  $P(p) \sim \bar{q}$ , and the fixed class containing the full cycles of  $P^2(p)$  and  $\bar{Q}(\bar{q})$ . Loops around these three trees form a simple obstruction for  $g$ . This obstruction is canonical by the Selinger characterization; see [char] or Theorem 3.7 in [quadJung]. So it is pinched according to the Canonical obstruction Theorem [Pilgrim, ext, quadJung], and marked points in the same ray-equivalence class will collide under the Thurston pullback  $\sigma_g$  of the formal mating. The issue of Theorem 4.1, and the underlying Theorem 3.11 in [quadJung], is that the colliding points do not wander.

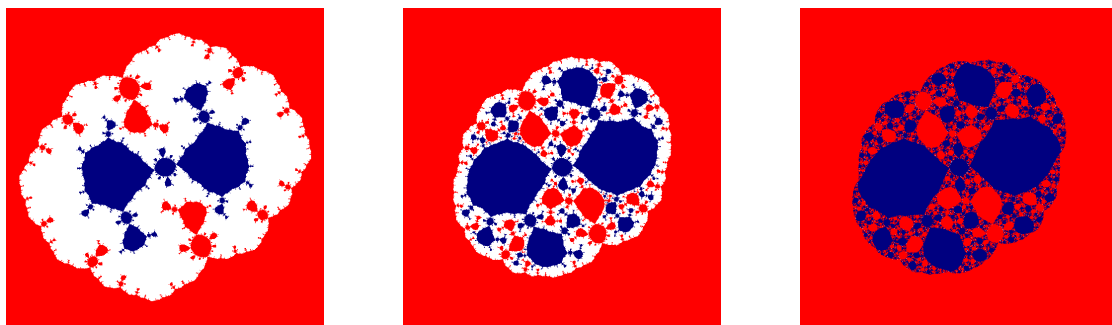
When the marked points are normalized such that the critical point of  $Q$  is at  $\infty$ , the polynomial fixed point within the fixed class is at  $-1$ , and its preimage at 1, then convergence is obvious: all marked points collide with  $\infty$ , 1, or  $-1$  as required. There is a subtle point, however, because this normalization is different from the usual normalization, where the fixed point on the equator of  $g$  is at 1. In the case of  $1/4 \amalg 1/2$  this is simply a rotation  $z \mapsto -z$ , but comparing the pullback for  $5/12 \amalg 1/6$  or  $3/28 \amalg 3/14$  in the two normalization means a variable rotation. So either we use the former normalization and get an additional marked point on the equator; then we must show it to converge to  $1 \pm i$  and conclude that the limit map is affine conjugate to the geometric mating in the usual normalization. Or take the usual normalization during the pullback and have three postcritical classes besides 1. So to show that convergence is independent of the normalization, the Canonical obstruction Theorem alone may be not sufficient. ■

## A Preview of anti-matings

Aa

While the simplest mating is  $f(z) = z^2$ , the prototypical anti-mating is given by  $f(z) = z^{-2}$ . Anti-mating is understood by gluing two filled Julia sets, which are mapped to each other by  $f$ . The notion of an anti-equator and anti-pseudo-equator was suggested by Daniel Meyer [questions, unmating]. Vladen Timorin has defined the formal anti-mating, and suggested a definition in terms of laminations in addition [extBound]. He attributed the idea to Douady–Hubbard; however, I have not been able to find it in [sdh] or the Orsay notes. There may be prior work in the thesis of Davood Ahmadi as well. The definitions and examples in the following sections provide a preview of results and ongoing research from [antiJung].

The basic idea is that the formal anti-mating  $g = P \sqcap Q$  and the topological anti-mating  $P \sqcup Q$  are quadratic maps, but the topology and dynamics is understood from the quartic polynomials  $Q \circ P$  and  $P \circ Q$ . Except in the symmetric case of  $P = Q$ , the quadratic dynamics of  $P(z) = z^2 + p$  and  $Q(z) = z^2 + q$  will be irrelevant. It turns out that the anti-matings with  $p = 0$  belong to the bitransitive family, while  $p = -q^2$  gives the critically 2-periodic family  $V_2$  [extBound]. See also the videos on [www.mndynamics.com](http://www.mndynamics.com).



Fabfour

**Figure 4:** An anti-mating giving a quadratic rational map with 0 4-periodic and  $\infty$  2-periodic. Note that the two quartic Julia sets show similar features from two attracting orbits. Left: formal anti-mating, middle: Thurston iteration, right: geometric anti-mating. The same map is obtained as the mating of Kokopelli and Basilica in a Figure of [CJ].

### A.1 Definitions and basic properties

Aad

For quadratic polynomials  $P(z) = z^2 + p$  and  $Q(z) = z^2 + q$ , the toy model is a self map  $\hat{g}$  of the disjoint union of two complex planes; the first is mapped to the second one with  $P$ , and the second to the first with  $Q$ . So  $\hat{g}^2$  maps both planes to themselves as  $Q \circ P$  and  $P \circ Q$ , respectively. Now:

- $\hat{g}$  has critical points at 0 in both planes. It is postcritically finite, if and only if  $(Q \circ P)^n(0)$  and  $(P \circ Q)^n(0)$  are finite.
- The filled Julia sets  $\mathcal{K}_{QP}$  and  $\mathcal{K}_{PQ}$  are connected, if and only if both critical orbits are bounded. They are mapped to each other by  $\hat{g}$ . The two quartic polynomials are semi-conjugate in both directions,  $P \circ (Q \circ P) = (P \circ Q) \circ P$  and  $Q \circ (P \circ Q) = (Q \circ P) \circ Q$ .

- The quartic Böttcher maps satisfy  $P = \Phi_{PQ}^{-1} \circ F \circ \Phi_{QP}$  and  $Q = \Phi_{QP}^{-1} \circ F \circ \Phi_{PQ}$  with  $F(z) = z^2$ .

Note that every normalized even quartic polynomial can be written as  $Q \circ P$  with unique  $p$  and  $q$ . The dynamics of  $\hat{g}$  is unrelated to  $P^n(0)$  and  $Q^n(0)$ . For a rotation by  $\zeta$  with  $\zeta^3 = 1$ , replacing  $p \rightarrow \zeta p$  and  $q \rightarrow \zeta^{-1} q$  means that  $Q \circ P$  is transformed with  $\zeta$  and  $P \circ Q$  with  $\zeta^{-1}$ . Consider the following definitions and properties:

- For  $\varphi_0 : \mathbb{C} \rightarrow \mathbb{D}$  and  $\varphi_\infty : \mathbb{C} \rightarrow \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  with suitable asymptotics at  $\infty$ , the formal anti-mating  $g = P \sqcap Q$  is defined by  $g(z) = z^{-2}$  on the equator, and  $\varphi_\infty \circ P \circ \varphi_0^{-1}$  and  $\varphi_0 \circ Q \circ \varphi_\infty^{-1}$  between the half-spheres.
- External rays are defined as  $\mathcal{R}(\theta) = \varphi_0(\mathcal{R}_{QP}(\theta)) \cup \varphi_\infty(\mathcal{R}_{PQ}(-\theta)) \cup \{\exp(i2\pi\theta)\}$ . Now  $g$  maps  $\mathcal{R}(\theta)$  to  $\mathcal{R}(-2\theta)$ .
- The essential anti-mating  $\tilde{g}$  is defined by collapsing postcritical ray-equivalence classes. If it is equivalent to a quadratic rational map, this defines the combinatorial anti-mating  $f$ . Probably it exists [TL], when there are no cyclic ray connections between fixed points of  $P \circ Q$  and  $Q \circ P$ . The rational maps from the Thurston Algorithm of  $g$  converge, when the orbifold of  $\tilde{g}$  is not  $(2, 2, 2, 2)$ .
- The topological anti-mating  $P \sqcup Q$  is defined by collapsing all ray-equivalence classes to points. In the postcritically finite case, it exists whenever the combinatorial anti-mating  $f$  exists, which is a geometric anti-mating in fact.
- Equipotential gluing is done with the quartic Böttcher maps. Slow mating is initialized analogously to the case of matings [quadJung], noting that the critical orbits are alternating between both half-spheres.

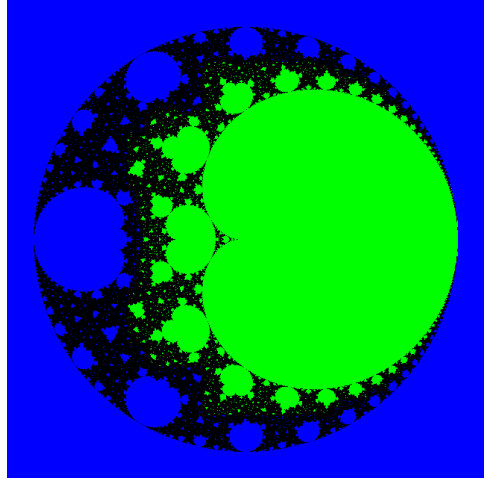
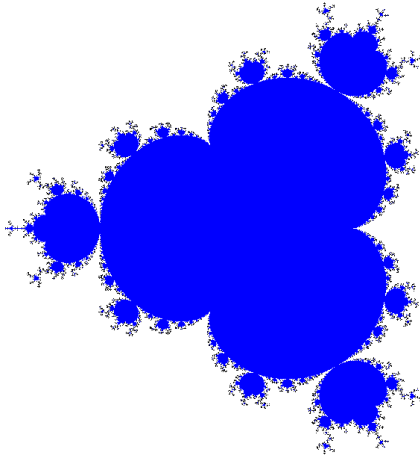
Unobstructed formal anti-matings correspond to rational maps having an anti-equator. While the parameter space of matings is  $\mathcal{M} \times \mathcal{M}$ , the parameter space of anti-matings consists of even quartic maps, which are not understood well combinatorially. So there is no explicit analog of non-conjugate limbs. Stronger results are obtained for specific one-parameter slices below.

## A.2 Bitransitive maps

Aab

Suppose  $p = 0$ , so  $Q \circ P(z) = z^4 + q$  and  $P \circ Q(z) = (z^2 + q)^2$ . The connectedness locus in the  $q$ -plane is the Multibrot set of degree 4. If the geometric anti-mating  $f$  of  $P(z) = z^2$  and  $Q(z) = z^2 + q$  exists, it is of the form  $f(z) = f_a(z) = 1 + \frac{a}{z^2}$ . These quadratic rational maps satisfy  $\dots \rightarrow 0 \Rightarrow \infty \Rightarrow 1 \rightarrow \dots$ ; when  $f_a$  is hyperbolic, it must be of bitransitive type, so it cannot be a mating. (Preperiodic matings in this family provide a simple case of mating discontinuity. See Theorem 3.9 in [raysJung].)

In the example of  $q^3 = -2$ , a 2-cycle of  $g$  is collapsed; convergence is obtained explicitly from the pullback relation of the Thurston Algorithm. The moduli space map is a Chebychev polynomial in fact. Note that it is not obvious that there are no cyclic ray connections, since both Julia sets are real-symmetric (or rotated), and that the orbifold of  $\tilde{g}$  is parabolic; the rational map  $f$  was discussed in Section 5.



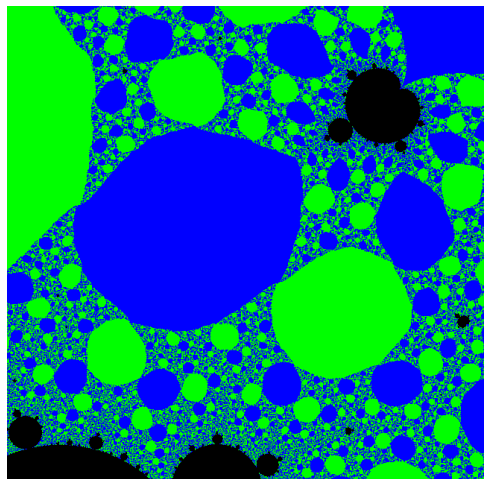
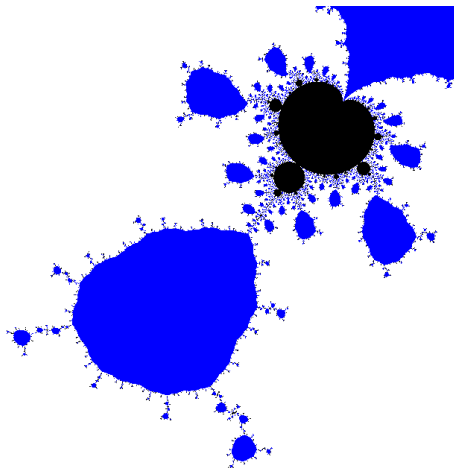
Flocia

**Figure 5:** Left: quartic polynomials  $P \circ Q$  or  $Q \circ P$  with  $P(z) = z^2$  and  $Q(z) = z^2 + q$ , the connectedness locus is the Multibrot set of  $z^4 + q$ . Right: rational maps  $f_a$  with  $0 \Rightarrow \infty$ . Probably the exterior disk with its blue and black sublimbs corresponds to one-third of the Multibrot set, or to that set parametrized by  $q^3$ .

### A.3 Basilica maps in $V_2$

Aa2

$p = -q^2$  gives  $Q \circ P(z) = (z^2 - q^2)^2 + q$  and  $P \circ Q(z) = z^4 + 2qz^2$ . This quartic family is discussed by Gamaliel Blé [talkCancun]. If the geometric anti-mating  $f$  of  $P(z) = z^2 - q^2$  and  $Q(z) = z^2 + q$  exists, it is in  $V_2$ :  $f(z) = f_b(z) = \frac{z^2 + b}{z^2 - 1}$  has a 2-periodic critical point  $\infty \Rightarrow 1 \rightarrow \infty$ . This example of anti-matings is due to Timorin [extBound] in a different normalization. All hyperbolic maps of disjoint type are known to be matings  $R \amalg B$  with the Basilica polynomial  $B(z) = z^2 - 1$  [...]. Some maps of disjoint type or capture type are anti-matings as well.



Flocib

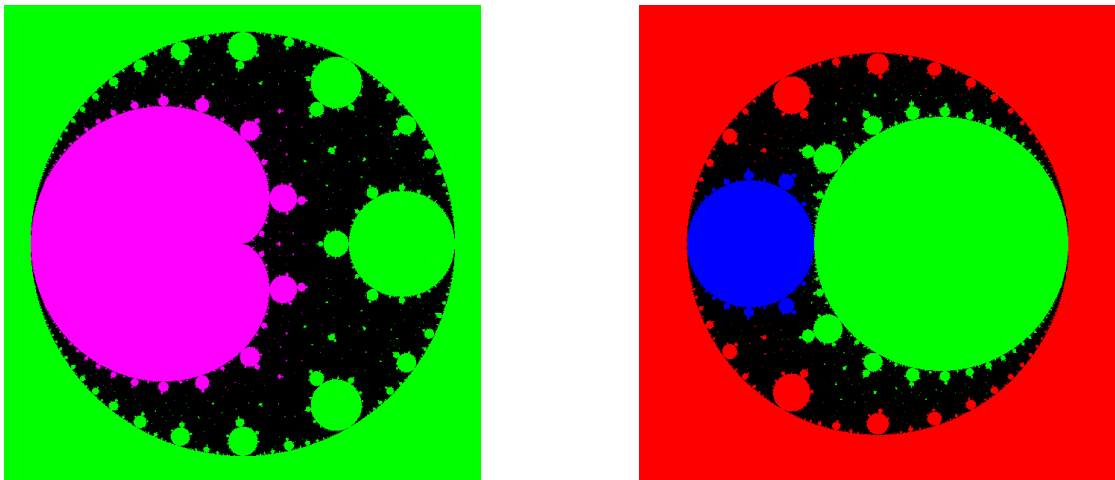
**Figure 6:** Corresponding subsets of the parameter planes: left quartic polynomials  $P \circ Q$  or  $Q \circ P$  with  $P(z) = z^2 - q^2$  and  $Q(z) = z^2 + q$ , right rational maps  $f_b$  with  $\infty$  2-periodic. Note that all capture components of the polynomial family are blue, while capture components of the rational family are blue or green.

## A.4 Symmetric maps and Chebychev matings

Quadratic rational maps are symmetric under inversion, when they are of the form  $f_c(z) = \frac{z^2+c}{1+cz^2}$ . Self-matings and self-anti-matings with  $P = Q$  are obviously symmetric, and in this case the quadratic dynamics of  $p \in \mathcal{M}$  is relevant. Now:

- When  $P$  is hyperbolic of period  $n$ , the mating  $P \amalg P$  has two disjoint critical orbits of period  $n$ . The same applies to the anti-mating  $P \amalg P$  if  $n$  is even. For odd  $n$ , it will be bitransitive of period  $2n$  instead.
- $f_c$  is the anti-mating of  $z^2 + p$  with itself, if and only if  $f_{1/c}$  is the mating of  $z^2 + p$  with itself. In particular, the anti-mating exists in the postcritically finite case, when  $p$  is not in the  $1/2$ -limb of  $\mathcal{M}$ .
- Then both  $f_c$  and  $f_{1/c}$  are semi-conjugate to the Chebychev-mating  $P \amalg T$  with  $T(z) = z^2 - 2$ . This transformation is due to Carsten Petersen [pasteMilnor].

Note that the symmetry locus contains hyperbolic maps as well, which are neither symmetric matings nor symmetric anti-matings. This applies to the mating of Airplane and Rabbit, e.g., and I do not know how this representation can be generalized. See the green–black locus in Figure 7 right. The corresponding maps in the Chebychev family have a description as matings with preperiodic polynomials from suitable limbs, which are partially shared; see Section 3.3 in [raysJung], and the green–black locus in Figure 7 left.



**Figure 7:** The right image shows the parameter plane of symmetric maps  $f_c$ . Conjecturally, the mating locus is blue–black and the anti-mating locus is red–black; each corresponds to the Mandelbrot set with its  $1/2$ -limb removed.

The Petersen transformation (Remark 3.4) maps both sets to the magenta–black subset of the left image, which is the locus of Chebychev matings within the family of rational maps  $f$ , such that  $f(\infty)$  is pre-fixed.