Small connectedness loci in parameter spaces

Wolf Jung, Gesamtschule Brand, Aachen, Germany.

1. Introduction and background
2. One-parameter slices in two-parameter families
3. Examples of small connectedness loci
4. Possible approaches toward a proof
5. Anti-mating
6. Generalized anti-mating
7. Conclusion

The images are made with Mandel, a program available from www.mndynamics.com.

Talk given at Workshop on Holomorphic Dynamics, Holbæk, October 21, 2023.

## 1a Motivation from a picture of $S_{4}$, courtesy Jack Milnor

These cubic polynomials $f(z)$ have a persistently 4-periodic critical point. Various hyperbolic components can be seen in the non-escape locus. Can we infer their type from the shape?

A decomposition of the Julia sets, for example by renormalization, is useful both for understanding the dynamics and topology of filled Julia
 sets $\mathcal{K}$, and for the structure of parameter spaces.

## 1b Recall quadratic polynomials

For $c \in \mathcal{M}$, the critical orbit of $f_{c}(z)=z^{2}+c$ is bounded and the filled Julia set $\mathcal{K}_{c}$ is connected. The images show the Mandelbrot set $\mathcal{M}$ and an example of $\mathcal{K}_{c}$, such that $z=0$ is 12 -periodic under $f_{c}$.


## 1c Renormalization and straightening

The renormalization $g_{c}(z)=f_{c}^{4}(z): U_{c}^{\prime} \rightarrow U_{c}$ is quadratic-like and can be straightened to another quadratic polynomial $f_{\hat{c}}$.
$\mathcal{M}$ contains a corresponding copy $\mathcal{M}_{4}$ of itself, with infinitely many decorations attached.


## 1d Straightening of quadratic-like families

Theorems by Adrien Douady and John Hamal Hubbard:

1) Suppose that $g_{c}(z)=f_{c}^{n}(z): U_{c}^{\prime} \rightarrow U_{c}$ is quadratic-like, the quasi-disks move holomorhically for $c \in U_{M}$, and $v_{c}-\omega_{c}$ winds once around 0 .
Then there is a straightening map $\chi: \mathcal{M}_{n} \rightarrow \mathcal{M}, c \mapsto \widehat{c}$. Its inverse is the tuning map $\widehat{c} \mapsto c=c_{n} * \widehat{c}$.
In the satellite case, sublimbs may be treated separately.
2) For every hyperbolic component $\Omega_{n} \subset \mathcal{M}$, there is a corresponding small Mandelbrot set $\mathcal{M}_{n}$, and their union is the locus of simple renormalization.

Note that to apply Theorem 1, the required domains may be constructed, for example, as explicit disks, from asymptotic formulas, or as puzzle-pieces.

## 2a Two-parameter families of rational maps

While quadratic polynomials form a one-parameter family, natural twoparameter families, which have two free critical points, are as follows:

Cubic polynomials have a superattracting fixed point at $\infty$ and two finite critical points. Up to affine conjugation they may be parametrized, for example, as $f(z)=z^{3}-3 a^{2} z+b$ or $f(z)=A\left(z^{3}-3 z\right)+B$.

Quadratic rational maps have two critical points. Up to Möbius conjugation they may be parametrized, for example, as $f(z)=\frac{z^{2}+A}{z^{2}+B}$ or $f(z)=c\left(z+\frac{1}{z}\right)+d$.

## 2b Types of hyperbolic maps and components

The Fatou components of a hyperbolic map with connected Julia set may be mapped as follows; this is used to classify hyperbolic components as well:


A - adjacent
$B$ - bitransitive


C - capture

D - disjoint


## 2c One-parameter families as one-dimensional slices

A critical relation like $f^{3}\left(\omega_{1}\right)=f^{7}\left(\omega_{2}\right)$ defines a one-parameter subfamily of cubic polynomials or quadratic rational maps, respectively. Here consider maps with an $n$-periodic critical point:
$S_{n}=\left\{f(z)=z^{3}-3 a^{2} z+b \mid z=-a\right.$ is $n$-periodic $\}$, $V_{n}=\left\{\left.f(z)=\frac{z^{2}+A}{z^{2}+B} \right\rvert\, z=\infty\right.$ is $n$-periodic $\}$.


## 2d Two families of even quartic polynomials

Any even quartic polynomial is a composition of two quadratic polynomials; interchanging these gives a semiconjugate quartic polynomial. Here consider quartic maps with a fixed critical point:
$(Q \circ P)(z)=\left(z^{2}-q^{2}\right)^{2}+q$, and
$(P \circ Q)(z)=\left(z^{2}+q\right)^{2}-q^{2}$.
The common connectedness locus is denoted by $\mathcal{B}$.


## 2e A model using two disjoint planes

The dynamics and combinatorics of $(Q \circ P)(z)=\left(z^{2}-q^{2}\right)^{2}+q$ and $(P \circ Q)(z)=\left(z^{2}+q\right)^{2}-q^{2}$ may be described by considering $P(z)=z^{2}-q^{2}$ and $Q(z)=z^{2}+q$ as maps between two different planes; compare the mapping sheme according to Jack Milnor:


Copies of $\mathcal{B}$ and of these Julia sets will be seen in the following examples; since anti-matings may be conjugated with a rotation, $V_{2}$ shows $(\mathcal{B})^{3}$ instead.

## 3a Straightening around a hyperbolic component

What small set is expected around a hyperbolic component?

| Type | two-parameter family | $S_{n}$ or $V_{n}$ |
| :--- | :--- | :--- |
| Adjacent | cubic connectedness locus | $S_{1}$ |
| Bitransitive | even quartic polynomials | $\mathcal{B}$ |
| Capture | $\left\{(c, z) \mid c \in \mathcal{M}, z \in \mathcal{K}_{c}\right\}$ | $\sim$ disk |
| Disjoint | $\mathcal{M} \times \mathcal{M}$ | $\mathcal{M}$ |

The two-dimensional straightening map of cubic polynomials was discussed by Hiroyuki Inou and Jan Kiwi; it is injective and it may be discontinuous at parabolic parameters, but the one-dimensional map shall be continuous.

Now if $f$ in $S_{n}$ or $V_{n}$ has a Fatou component of type $A$ or $B$, the renormalization $f_{c}^{n}: U^{\prime} \rightarrow U$ has a fixed critical point, so its straightening will be in $S_{1}$ or $\mathcal{B}$, respectively. It remains to show that the renormalization locus gives all of $S_{1}$ or $\mathcal{B}$.

## 3b Example: $S_{2}$

The image of $S_{2}$ shows two hyperbolic components with adjacent dynamics and two with bitransitive dynamics; actually the parametrization is a two-fold cover of moduli space.


## 3c Example: $S_{3}$

In $S_{3}$ there are four hyperbolic components with adjacent dynamics and eight of bitransitive type.


By the way, here is a zoom video around a Misiurewicz point: [misiS3].

## 3d Eample: $V_{3}$

Quadratic rational maps in $V_{n}$ do not have hyperbolic components of adjacent or escaping type. In $V_{3}$ there are two components with bitransitive dynamics. (The Julia sets and dynamics will be discussed later on p. 6b.)


## 3e Eample: $V_{4}$

In $V_{4}$ there are six bitransitive hyperbolic components.


On the right hand side, copies of $\mathcal{K}_{Q \circ P}$ contain green or blue disks, while the unique copy of $\mathcal{K}_{P \circ Q}$ around $\infty$ has red disks.

## 3f Example: $V_{5}$

This parametrization of $V_{5}$ contains the square-root of a quartic polynomial, so there are two sheets of parameter space; both contain bitransitve components, and some are primitive.


## 4a Conjectural description

For any bitransitive hyperbolic component $\Omega$ in $S_{n}$ or $V_{n}$, I expect a surrounding copy of $\mathcal{B}$ in parameter space.
For $V_{n}$, a big limb may be missing, and in $V_{2}$ we have $(\mathcal{B})^{3}$.
In the dynamic plane, we can find copies of $\mathcal{K}_{Q \circ P}$ and $\mathcal{K}_{P \circ Q}$ replacing hyperbolic components.

What does that mean and how can we prove it?

## 4b Approach with renormalization

Analogously to p. 1d, we may try to construct an analytic family of quartic-like restrictions of the first-return map, or a pair of quadratic maps.

This works well in certain cases, but in general there will be two problems:
The small Julia sets may have common boundary points; then the disks $U_{c}^{\prime} \rightarrow U_{c}$ may be hard to define or to control. In the case of $V_{n}$, there may be infinitely many common boundary points in fact.

Even in $S_{n}$, in the primitive case, we need to know more about landing properties of external rays, to define define these disks as puzzle-pieces.

## 4c Approach with Thurston theory

In the postcritically finite case, we may construct (or decompose) a map as follows:
Define a branched covering $g: S^{2} \rightarrow S^{2}$, which need not be holomorphic; it shall have the desired combinatorics of critical orbits.
Choose a homeomorphism $\varphi_{0}: S^{2} \rightarrow \widehat{\mathbb{C}}$ and pull it back with a sequence of rational maps $f_{n}$ according to the commuting diagram.
Under suitable assumptions, the rational maps converge to the desired one, and the homeomorphisms converge up to homotopy (relative to the marked points).
Actually, the pullback map is defined on a Teichmüller space.


## 4d Comparison

| Renormalization and straigthening | Thurston theory |
| :--- | :--- |
| may be postcritically infinite | mostly postcritically finite |
| continuous dependence on <br> parameters | continuity is not proved |
| straightening by quasi-conformal <br> surgery, tuning is defined as the in- <br> verse of straightening | both directions are possible |
| fairly easy theory, no numerics so <br> far, but combinatorial description | simple numerical implementation |
| Fatou components shall touch only <br> finitely often | Fatou components may touch <br> infinitely often |

## 5a Anti-mating

Anti-mating or cross-mating $f \simeq P \prod Q$ is a special case of the previous discussions, starting with the bitransitive component around $f(z)=1 / z^{2}$. And it can be defined analogously to mating, starting with a formal antimating $P \sqcap Q$ and collapsing postcritical ray-equivalence classes.
When $p=-q^{2}$, the resulting map is in $V_{2}$. See a phase space video [mateK-Banti] and a deformation video [slowanti2]. Note that the big limb of $(\mathcal{B})^{3}$ is missing. For $\partial \mathcal{B}$, this construction goes back to Vladlen Timorin.


## 5b Properties of anti-matings

Using a definition in terms of laminations, Davoud Ahmadi Dastjerdi has shown that a formal anti-mating $P \sqcap Q$ has only removable obstructions, whenever the three fixed rays land at different fixed points. So there is a corresponding rational map $f$, which is a combinatorial anti-mating and a geometric antimating, with a possible exception for type ( $2,2,2,2$ ).

We also have the mating $f^{2} \simeq(Q \circ P) \coprod(P \circ Q)$, so why do we need antimatings? Staying in the quadratic family may simplify understanding the combinatorics and dynamics, and we may use the Levy cycle criterion. Anti-matings without postcritical identifications, especially hyperbolic anti-matings, are characterized by an anti-equator; this was shown independently by Ma Liangang.

Since the two Julia sets share all of their boundaries, anti-matings cannot be constructed by renormalization, and only geometrically finite maps can be treated directly with Thurston theory.

## 6a Generalizations of anti-mating

Starting from the 3-periodic bitransitive map $f(z)=1-\frac{1}{z^{2}}$ in $V_{3}$, there is a construction analogous to anti-mating. The relevant Fatou components share infinitely many points. Probably the only non-removable obstruction happens when $1 / 3$ and $2 / 3$ land together, so $\mathcal{B}$ loses only one of the three big limbs.


## 6b Bitransitive components in $V_{3}$

For the example of $V_{3}$ from page 3 d , the sketch shows how $\mathcal{K}_{P \circ Q}$ and $\mathcal{K}_{Q \circ P}$ shall appear within the three periodic components.
[A deformation video is under construction.]


## 6c Small Multibrot sets in persistently bitransitive families

When anti-mating is performed with $P(z)=z^{2}$ and $Q(z)=z^{2}+q$, the relevant polynomial connectedness locus is the Multibrot set $\mathcal{M}_{4}$ for $(Q \circ P)(z)=$ $z^{4}+q$ and $(P \circ Q)(z)=\left(z^{2}+q\right)^{2}$. The resulting rational map is of the form $f_{a}(z)=1+\frac{a}{z^{2}}$ with $f_{a}(0)=\infty$, and we see a copy of $\left(\mathcal{M}_{4}\right)^{3}$ around the outer component of period 2. See also the approach of Pascale Roesch and Bastien Rossetti, which uses the mating $f_{a}^{2} \simeq(Q \circ P) \coprod(P \circ Q)$ and a continuation to the molecule along satellite components. A deformation video [slowanti0].


## 6d Anti-mating for a persistently bitransitive family

Analogously to the discussion of $V_{3}$, other parts of parameter space can be described by starting from the 3-periodic $f(z)=1-\frac{1}{z^{2}}$.
The images show the Julia sets for $\left(z^{2}\right) \prod\left(z^{2}-1\right)$ of period 4 and the generalized anti-mating of period 6 .
[A deformation video is under construction.]


## 7 Conclusion

$S_{n}$ and $V_{n}$ seem to contain small copies of $\mathcal{B}$, and also $S_{1}$ in the former case. These are recognizable from their shapes.
Partial proofs may be based on renormalization or on Thurston theory. A complete proof for all $n$ will require a deeper understanding of the combinatorics, for example landing properties of external rays or realizability of laminations.

The images show $\mathcal{B}$ again, and subsets of $S_{3}$ and $V_{5}$.


Hjertelig tak!

