Local similarity between the Mandelbrot set and Julia sets.

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$f_c(z) = z^2 + c$, $\mathcal{K}_c = \{z \,|\, f_c^n(z) \not\to \infty\}$, $\mathcal{M} = \{c \,|\, c \in \mathcal{K}_c\}$.

An example of local similarity was given by Peitgen 1988:
after an affine rescaling, the decorations of a little Mandelbrot set
and a little Julia set "look the same".

The images were made with Mandel, a program available from www.mndynamics.com.

Intuition of local similarity

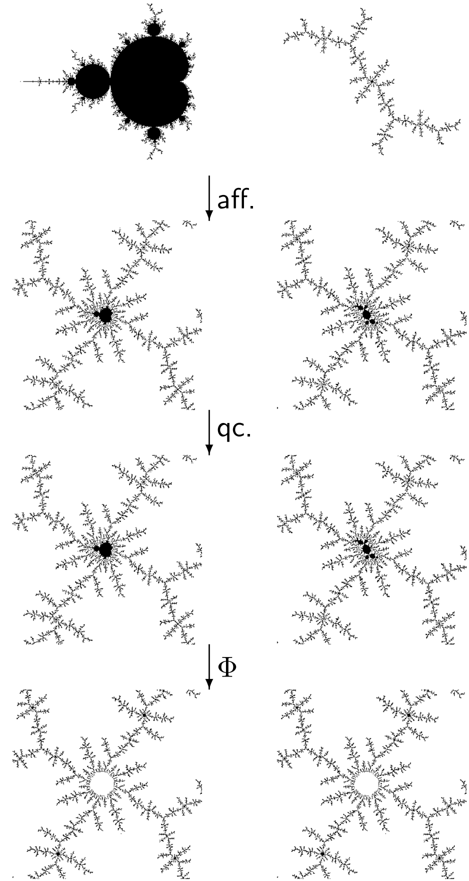
Observations: $\mathcal{M}$ and $\mathcal{K}_c$ are looking similar (small Hausdorff distance). This similarity is lost both very near and very far away.
The similarity is between the decorations.

1st idea:

parameter c is in decoration of little $\mathcal{M}$
$\Leftrightarrow$ crit. value c is in decoration of little $\mathcal{K}$.
Aim: control the changes of the dynamic decorations. Mostly bending, because preimages of little $\mathcal{K}$ are very small.

2nd idea: Control is needed only in a large fundamental annulus, then pull it back.
Boettcher Φ makes this explicit, gives the bending.

Main

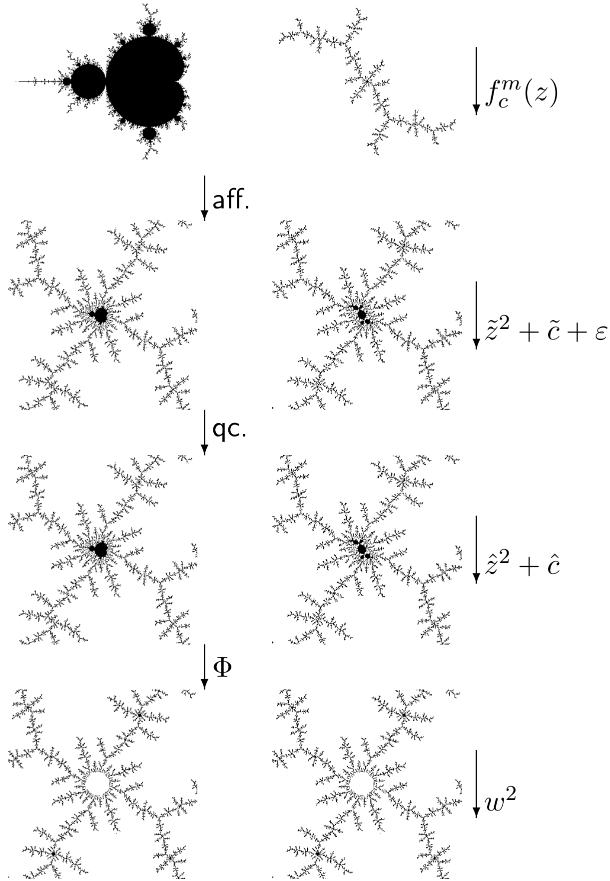


Formulation of local similarity

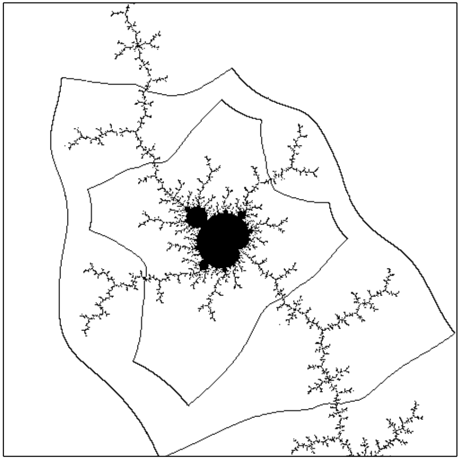
Start with affine conjugation from $f_c^m(z)$ to $\tilde{z}^2 + \tilde{c} + \varepsilon$ on large disk. Then a quasi-conformal conjugation, close to the identity, gives $\hat{z}^2 + \hat{c}$. For $\hat{c}$ in little $\mathcal{M}$:

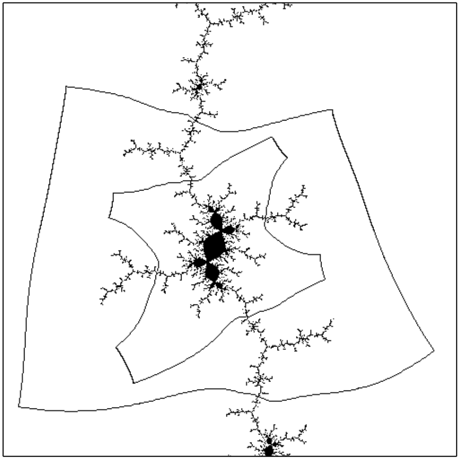
1) On a large disk, the decorations satisfy $\Phi_M(\hat{\mathcal{M}}) \approx \Phi_{\hat{c}}(\hat{\mathcal{K}}_{\hat{c}})$.
2) $\tilde{\mathcal{M}} + \frac{1}{2} \approx \tilde{\mathcal{K}}_{\tilde{c}}$ for $|\tilde{c}| > 10$.
3) $\mathcal{M} - c' \approx \lambda(\mathcal{K}_c - c)$.

Theorem 0 for the centers c_n at a Misiurewicz point a, $c_n \sim a + K\rho_a^{-n}$: the relative distance in 1) goes to 0 on a radius $\mathcal{O}(\rho_a^{n/4})$ as $n \to \infty$. Main



Renormalization: basic ideas





Quadratic-like mapping: smaller disk is mapped 2:1 to larger disk.

Little Julia set: not escaping — other points escaping through the annulus.

Either escaping to ∞, if not in $\mathcal{K}_c$. Or in decoration: mapped to a distant part of $\mathcal{K}_c$, maybe returning later. Attached to preimages of little β.

Little Mandelbrot set and decorations attached to little β-Misiurewicz points.

The annulus and its preimages are scaled as ρ^{-n}, $\rho^{-\frac{3}{2}n}$, $\rho^{-\frac{7}{4}n}$, $\rho^{-\frac{15}{8}n}$, ... Main

Renormalization with a large annulus

Dynamics of $f_c(z)$ for $c \approx c_n$, $z \approx \omega_c$:

1) mapped close to repelling periodic point by conformal $f_c^k(z)$
2) mapped n times around the repelling periodic point
3) then close to 0, mapped back 2:1

Using the Koenigs conjugation ϕ_c and the multiplier ρ_c, this gives

$f_c^{np+k+1}(z) = K_1 \rho_a^{2n}(z - \omega_c)^2 + c + \mathcal{O}(\rho_a^{-\frac{7}{4}n})$

for $z - \omega_c = \mathcal{O}(\rho_a^{-\frac{19}{12}n})$, $c - c_n = \mathcal{O}(\rho_a^{-\frac{7}{4}n})$ [E–E, D–H, McM].

The affine rescaling $\tilde{z} = K_1 \rho_a^{2n}(z - \omega_c)$ and $\tilde{c} = K_2 \rho_a^{2n}(c - c_n)$ gives $\tilde{z}^2 + \tilde{c} + \mathcal{O}(\rho_a^{n/4})$ for $\tilde{z} = \mathcal{O}(\rho_a^{\frac{5}{12}n})$, $\tilde{c} = \mathcal{O}(\rho_a^{n/4})$.

Lemma 1: The Straightening Thm. gives a quasi-conformal conjugation to $\hat{z}^2 + \hat{c}$. By explicit construction we have $\hat{z} - \tilde{z} = \mathcal{O}(\rho_a^{-n/6})$ and $\hat{c} - \tilde{c} = \mathcal{O}(\rho_a^{-n/6})$.

Main

Asymptotic models: The Koenigs conjugation ϕ_c at the repelling periodic point is maping $\mathcal{K}_c$ to the asymptotic model X_c, which is linearly self-similar with the scaling factor ρ_c. Now $\mathcal{M}$ is asymptotically self-similar at a [Tan Lei]: $\rho_a^n(\mathcal{M} - a) \to K_0 X_a$ in Hausdorff-Chabauty metric. By the same techniques:

$$\rho_a^n(\mathcal{M} - c_n) \to K_0(X_a - \phi_a(0))$$
$$\rho_a^{\frac{3}{2}n}(\mathcal{M} - c_n) \to K_1(X_a - \phi_a(0))^{1/2}$$
$$\rho_a^{\frac{7}{4}n}(\mathcal{M} - c_n) \to K_2(X_a - \phi_a(0))^{1/4}$$

There are analogous asymptotics for the Julia set. In both planes, there are m decorations of length $\sim \rho_a^{-n}$, $2m$ of length $\sim \rho_a^{-\frac{3}{2}n}$, $4m$ of length $\sim \rho_a^{-\frac{7}{4}n}$ ...

Lemma 2: Two asymptotic models are combined in the fundamental annulus $R'|\rho_a|^{-\frac{19}{12}n} \le |z - \omega_c| \le R|\rho_a|^{-\frac{7}{6}n}$. For $c = c_n + \mathcal{O}(\rho_a^{-\frac{7}{4}n})$, this part of $\mathcal{K}_c$ is contained in a relative ε-neighborhood of $\mathcal{K}_{c_n}$, with $\varepsilon \to 0$ as $n \to \infty$.

The relative distance estimate is transfered to $\tilde{z}$ and $\hat{z}$, and pulled back using the Boettcher conjugation $\Phi_{\hat{c}}$. Control of dynamic decorations gives control of parameter decorations. Proves Theorem 0.

Main

Hairiness

Smallest possible scale: $c - c_n$ and $z - c_n$ are $\mathcal{O}(\rho_a^{-2n})$, $\tilde{c}$ and $\tilde{z}$ are $\mathcal{O}(1)$. Then the (absolute) Hausdorff distance of dynamic decorations and parameter decorations $\to 0$ as $n \to \infty$. For extremely large n, they will look as follows:

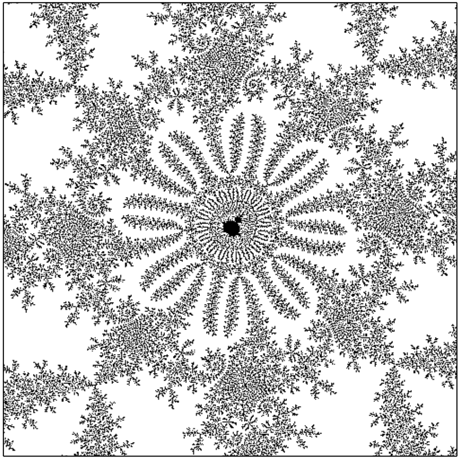
1) All decorations of the little $\hat{\mathcal{M}}$ or $\hat{\mathcal{K}}_{\hat{c}}$ are converging to binary rays, if $a \in \mathbb{R}$.

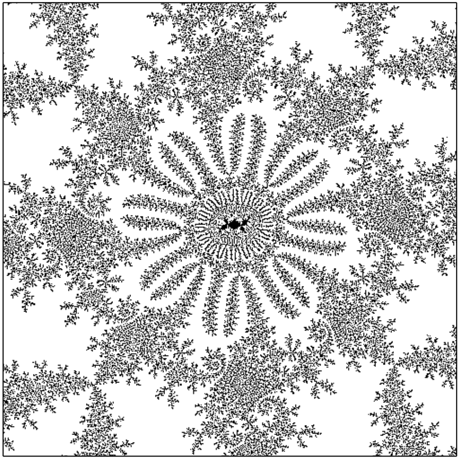
2) At least for $a = -2$, the area of the decorations $\to 0$.

Probably true for all Misiurewicz points $a \in \mathcal{M}$, when rays are replaced with $\Phi^{-1}(\rho_a^{\mathbb{R}})$ for a suitable branch of $\log \rho_a$.

Main

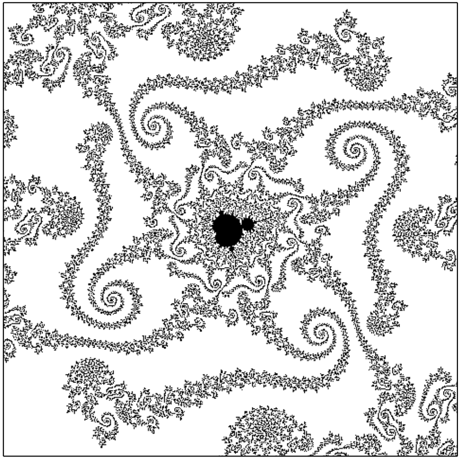
Example 0

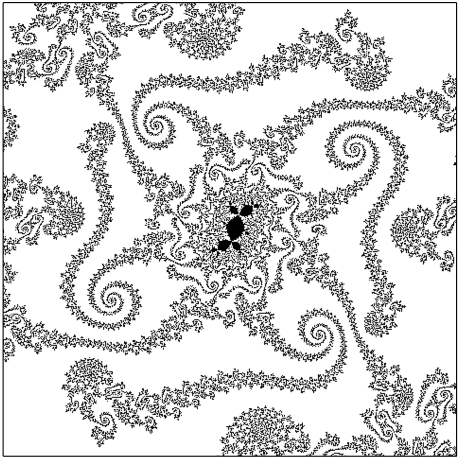




Main, Example 1, 2, 3, 4, 5, 6, 7.

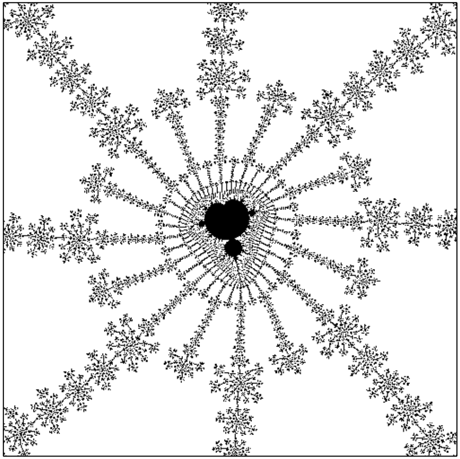
Example 1

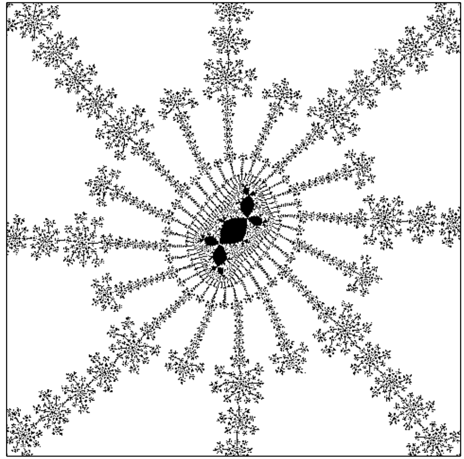




Main, Example 1, 2, 3, 4, 5, 6, 7.

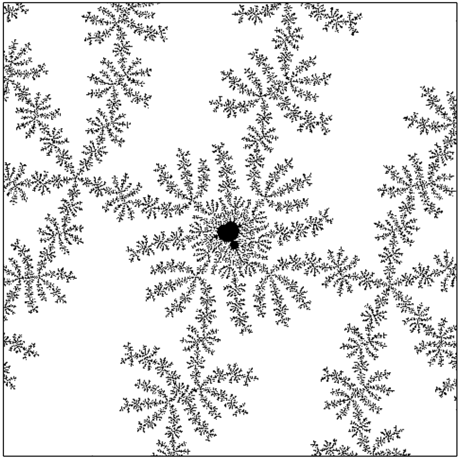
Example 2

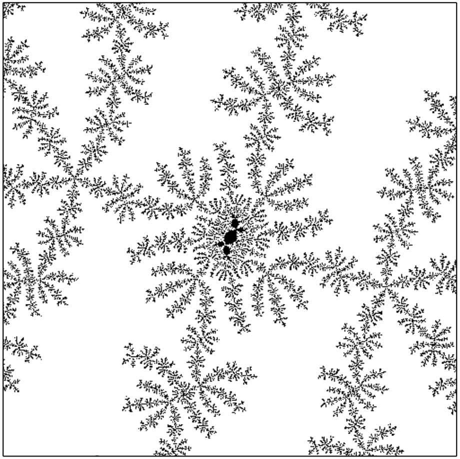




Main, Example 1, 2, 3, 4, 5, 6, 7.

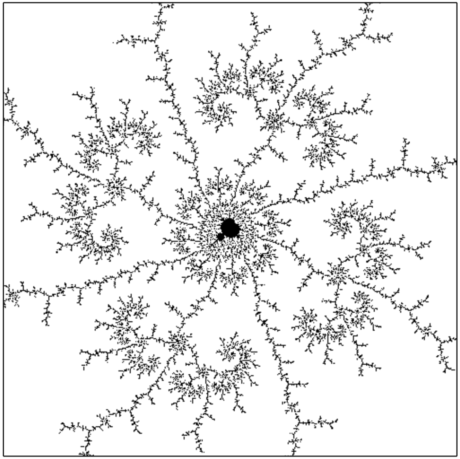
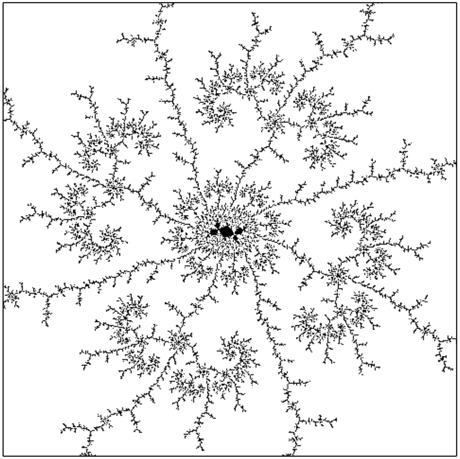
Example 3





Main, Example 1, 2, 3, 4, 5, 6, 7.

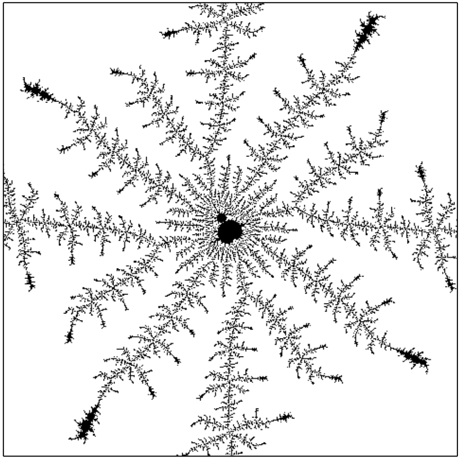
Example 4

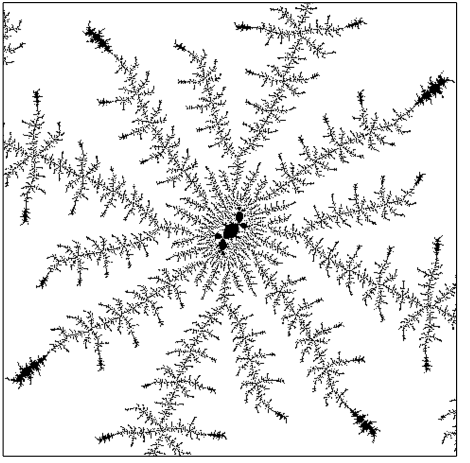
 

Close to a root: the relative error seems to be small but does not go to 0.

Main, Example 1, 2, 3, 4, 5, 6, 7.

Example 5

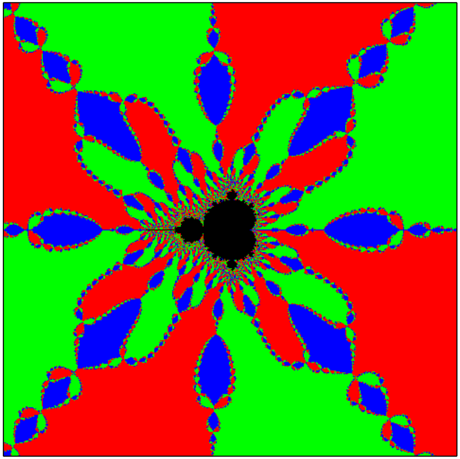
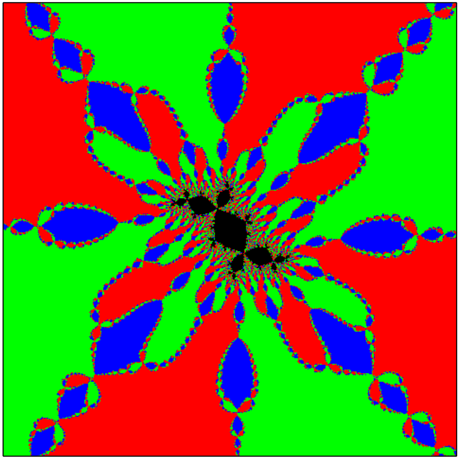




From the family of cubic polynomials with a persistent Siegel disk.

Main, Example 1, 2, 3, 4, 5, 6, 7.

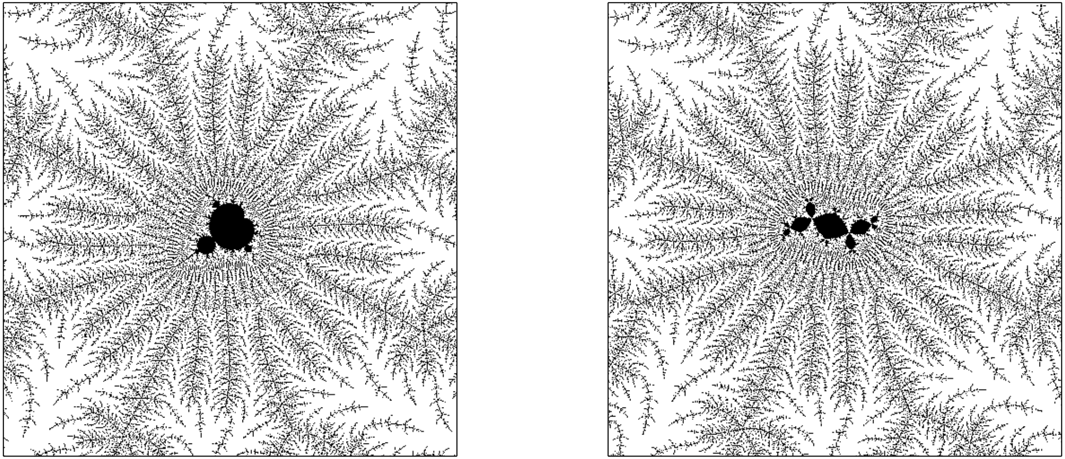
Example 6

From the cubic Newton family.

Main, Example 1, 2, 3, 4, 5, 6, 7.

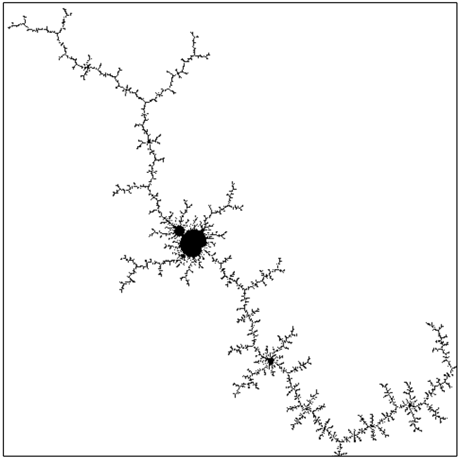
Example 7

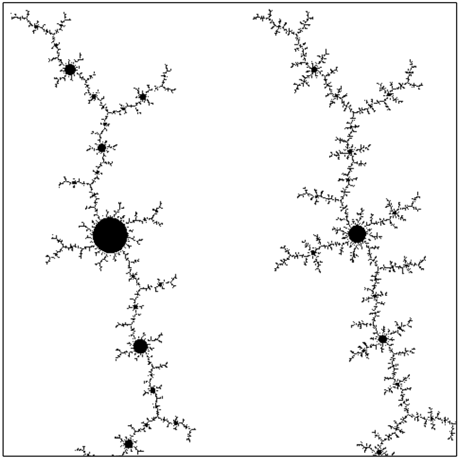


From the family $f_c(z) = c \sin z$.

Main, Example 1, 2, 3, 4, 5, 6, 7.

Appendix: Comparison of $\mathcal{M}$ and different $\mathcal{K}_c$





Main