# Some Explicit Formulas for the Iteration of Rational Functions

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**Abstract**

Let $f$ be a rational function, which has $k$ $n$-cycles under iteration. By using the symmetry of the underlying equation of degree $k \cdot n$, it is reduced to equations of degree $k$ and $n$. This is explained in terms of Galois theory.
The 3- and 4-cycles of $f_c(z) = z^2 + c$ are obtained explicitly. This yields the corresponding multiplier, which maps hyperbolic components of the Mandelbrot set conformally onto the unit disk.

## 1 Introduction

For a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, denote the $n$-th iterate by $f^n$. $z_0$ is in the Julia set of $f$, if the sequence $(f^n(z))$ is not normal in any neighborhood of $z_0$. We consider the family of quadratic polynomials $f_c(z) = z^2 + c$. The Mandelbrot set $\mathcal{M}$ contains those parameters $c \in \mathbb{C}$, such that the Julia set of $f_c$ is connected, and for $c \notin \mathcal{M}$, the corresponding Julia set is a Cantor set. Since 0 is the only critical point of the polynomial $f_c$, $c \in \mathcal{M}$ iff the orbit $(f_c^n(0))$ is bounded [4, p. 124]. This is used to obtain computer images of $\mathcal{M}$.

A $n$-cycle of $f_c$ consists of distinct points $z_1 \dots z_n$ with $f_c(z_1) = z_2, \dots,$
$f_c(z_n) = z_1$. The corresponding multiplier is $\lambda = f_c^{n\prime}(z_1) = 2^n z_1 \cdot z_2 \cdots z_n$. The cycle is attracting, if $|\lambda| < 1$. The set of those $c$, such that $f_c$ has an attracting $n$-cycle, is a union of components of the interior of $\mathcal{M}$, which are called hyperbolic. These are mapped conformally onto the unit disk by $\lambda$. It is well known that $\lambda$ is an algebraic function, with $(\lambda/2)^2 - \lambda/2 + c = 0$ for $n = 1$ and $\lambda = 4(c + 1)$ for $n = 2$ [12, p. 161].

We describe an algorithm to obtain these functions for every $n$, and give the results for period 3 and 4. Define the polynomials $g_n(z,\, c)$ recursively by

$$f_c^n(z) - z = \prod_{d|n} g_d(z,\, c) \ , \tag{1}$$

then the zeros of $g_n$ are the $n$-periodic points of $f_c$. For $n \geq 3$, the degree of $g_n$ is at least 6. In general, only polynomial equations of degree 4 or less can be solved explicitly, but $g_n$ satisfies the symmetry relation $g_n(z,\, c) = 0 \Rightarrow g_n(f_c(z),\, c) = 0$, which is used to reduce the equation. The resulting algorithm is best understood in terms of Galois theory.

## 2 The cycles of $f_c$

Except for some values of $c$, at which a bifurcation occurs, $g_n$ has $k \cdot n$ simple zeros. These form $k$ $n$-cycles $z_1^{(j)} \ldots z_n^{(j)}$ with $f_c(z_i^{(j)}) = z_l^{(j)}$, $l = i{+}1 \pmod n$. This suggests the following algorithm: Define $s_n(z,\, c,\, a) = z + f_c(z) + \ldots + f_c^{n-1}(z) - a$. Then $s_n(z_i^{(j)},\, c,\, a) = s_n(z_l^{(j)},\, c,\, a)$, thus $a$ can be chosen such that the greatest common divisor of $g_n$ and $s_n$ is of degree $n$ (namely $a = a^{(j)} = z_1^{(j)} + \ldots + z_n^{(j)}$). We perform Euklid's algorithm with $g_n$ and $s_n$. The remainder with degree $< n$ must vanish. This yields an equation $h_n(a,\, c) = 0$, where $h_n$ is the g.c.d. of the coefficients of this remainder. Denote the remainder of degree $n$ by $j_n(z,\, c,\, a)$. It is the g.c.d. of $g_n$ and $s_n$, if $a$ satisfies $h_n(a,\, c) = 0$. We have applied this algorithm to $g_3$, $g_4$ and $g_5$ and give the results for $n = 3,\, 4$ in the following

**Theorem 1**
(Netto) *Determine $h_n$ and $j_n$ from the algorithm described above. The $n$-cycles of $f_c$ are obtained by solving $h_n(a,\, c) = 0$ for $a$ and then $j_n(z,\, c,\, a) = 0$ for $z$. This can be done explicitly for $n = 3,\, 4$. We have*

$$\begin{aligned}
h_3(a,\, c) &= a^2 + a + c + 2 \\
j_3(z,\, c,\, a) &= z^3 - a\, z^2 + (-a + c - 1)\, z - (ac + c + 1) \\
h_4(a,\, c) &= a^3 + (4c + 3)\, a + 4 \\
j_4(z,\, c,\, a) &= \Big(4az^2 - 2a^2z - a^2 - a - 4\Big)^2 - a^2(a^2 + 4)\Big(2z - a - 1\Big)^2 \ .
\end{aligned}$$

All of these formulas can be translated to the logistic map $x \mapsto A\, x\, (1 - x)$. These formulas have been derived a hundred years ago by Netto in [11]. He started with the problem of finding a polynomial with a cyclic Galois group, and arrived at the iteration of rational functions. Brown [2] has given formulas for $n = 5$ and $n = 6$, and these questions have also been addressed in [13], [14], [8]. The method of these authors consists of eliminations with the coefficients of the polynomials, by using relations like

$$\sum_i \Big(z_i^{(j)}\Big)^2 = \sum_i \Big(z_{i+1}^{(j)} - c\Big) = a^{(j)} - nc \ , \tag{2}$$

and thus it is shown that the coefficients of $j_n$ (w.r.t. $z$) are rational functions of $c$ and $a$.

Our method using Euklid's algorithm proposed above seems to be simpler and is easily performed by Maple (see the last page). For $n > 5$, however, the required memory and time grow immensely.

An equation of degree 3 can be solved with Cardano's formula, and an equation of degree 4 is reduced to two quadratic equations, after solving the associated cubic resolvent. We will see in Section 4 that the Galois group of $j_n$ is cyclic. This implies that the discriminant of $j_3$ is a square, namely $(4a^2 + 6a + 9)^2$, and the resolvent of $j_4$ is reducible [10, p. 126], which has led us to the simplified form of $j_4$ given above. Thus solving $j_4(z, c, a) = 0$ is reduced to

$$4az^2 - 2a^2z - a^2 - a - 4 = \pm\, a\sqrt{a^2 + 4}\left(2z - a - 1\right) . \tag{3}$$

# 3 Formulas for components of the Mandelbrot set

From $j_n$, the multiplier $\lambda = 2^n\mu$ with $\mu = z_1 \cdot z_2 \cdots z_n$ is obtained by using Vieta's theorem. This yields

**Theorem 2**

(Stephenson) *For a hyperbolic component $\mathcal{H}$ of $\mathcal{M}$ corresponding to attracting $n$-cycles of $f_c$, the multiplier $\lambda : \mathcal{H} \to D_1(0)$ is a suitable branch of $k_n(c, \mu) = 0$, where $\lambda = 2^n\mu$ and the polynomial $k_n$ is obtained by eliminating $a$ from the equations*

$$h_n(a, c) = 0 \qquad \textit{and} \qquad \mu = (-1)^n \, \frac{\text{trailing coefficient}\,(j_n)}{\text{leading coefficient}\,(j_n)} \; .$$

*For $n = 3$ and $n = 4$ this yields*

$$\begin{aligned} k_3(c, \mu) &= c^3 + 2\,c^2 + (1-\mu)\,c + (1-\mu)^2 \\ k_4(c, \mu) &= c^6 + 3\,c^5 + (\mu+3)\,c^4 + (\mu+3)\,c^3 \\ &+ (2-\mu-\mu^2)\,c^2 + (1-\mu)^3 \; . \end{aligned}$$

$k_n$ has been given in [13] for $n = 3, 4, 5$, in [14] for $n = 6$ and in [15] for $n = 7$. In the latter case, a numerical method is used. $k_n$ has degree $k$ with respect to $\mu$ and degree $nk/2$ w.r.t. $c$. For $n = 3$ or 4, $\lambda$ is obtained explicitly from $c$. For $n = 3$, $c$ is obtained from $\lambda$, and the boundary of the 3 corresponding hyperbolic components of $\mathcal{M}$ is determined by $|c + 2 \pm c\sqrt{-4c - 7}| = 1/4$. In $\mathbb{R}^2$, this is a curve of order 12. These formulas can be used to draw more accurate computer images of the Mandelbrot set, and to determine, e.g., the points of bifurcation from period 3 to period $m \cdot 3$, where $\lambda^m = 1$.

In principle, $k_n$ can be obtained without employing the results of Theorem 1, by eliminating $z$ from $z \cdot f_c(z) \cdots f_c^{n-1}(z) - \mu = 0$ and $g_n(z, c) = 0$, but this does not provide a simplification, since the degrees are increased and the computation requires even more steps.

# 4 The Galois group of $g_n$

The algorithm of Theorem 1 can be understood in terms of Galois theory as follows: The two basic ideas of Galois theory are to consider the problem of determining the roots of a polynomial as a problem of field extensions, and to translate this to the investigation of a finite group. Denote the field $\mathbb{Q}(c)$ by $K$ and the splitting field of $g_n \in K[z]$ by $L$. The Galois group $G$ consists of those automorphisms of $L$, which are leaving $K$ fixed. It is represented by permutations of the zeros of $g_n$. For $\alpha \in G$, we have $f_c(z \cdot \alpha) = f_c(z) \cdot \alpha$. Since $f_c$ acts on the zeros as the permutation $\alpha_0 = (z_1^{(1)} \dots z_n^{(1)}) \dots (z_1^{(k)} \dots z_n^{(k)})$, $G$ must be contained in the centralizer of $\alpha_0$ in the symmetric group $S_{kn}$, which is a wreath-product $S_k \wr C_n$. We have $S_k \wr C_n \cong S_k \times (C_n \otimes \dots \otimes C_n)$, where $S_k$ is permuting the different cycles of $g_n$, while each $C_n$ is acting on the elements of a unique cycle. Now the algorithm of Section 2 corresponds to the normal series

$$S_k \wr C_n \trianglerighteq (C_n \otimes \dots \otimes C_n) \trianglerighteq \dots \trianglerighteq C_n \otimes C_n \trianglerighteq C_n \trianglerighteq 1 \ .$$

The Galois group of $h_n \in K[z]$ is contained in $S_k$, and for $h_n(a, c) = 0$ the Galois group of $j_n \in K(a)[z]$ is cyclic of order $\le n$. Thus $j_n = 0$ can always be solved explicitly, but $h_n = 0$ is in general not solvable for $k > 4$. Up to now, we have shown that $G \le S_k \wr C_n$. In the case of $n = 3$ or $n = 4$, the formulas of Theorem 1 show that $G \cong S_k \wr C_n$ in general, i.e. if $c$ is transcendental, or equivalently, if $\mathbb{Q}(c)$ is understood as the field of rational functions in one variable.

**Theorem 3**
(Bousch) *The Galois Group of* $g_n(z, c) \in \Big(\mathbb{C}(c)\Big)[z]$ *is isomorphic to* $S_k \wr C_n$.

In [1] the manifold given by $g_n(z, c) = 0$ is considered as a covering of the parameter plane (with the bifurcation points removed). Bousch shows that the fundamental group of the punctured parameter plane acts on the fibers as $S_k \wr C_n$, and this action is isomorphic to the Galois group of $g_n$ [6]. A similar proof is given in [9], which extends to $z^d + c$.

# 5 Summary and generalization

If $f$ is a rational function with $k$ $n$-cycles, the underlying equation of degree $k \cdot n$ is reduced to one equation of degree $k$ and $k$ equations of degree $n$. The first is solvable explicitly at least if $k \le 4$, while the latter equations are always solvable, as Galois theory shows.

Usually, $s_n(z, a) = z + f(z) + \dots + f^{n-1}(z) - a$ will work, but if e.g. $a^{(1)} = a^{(2)}$ as for $f(z) = z^2 - 4/3$ with $n = 5$, then $s_n$ must be replaced by some higher-degree symmetric polynomial.

If $f$ is not a polynomial, $g_n$ and $s_n$ must be understood as the numerators of certain rational functions. As an example, consider $f(z) = z - \frac{z^3-1}{3z^2}$, which arises when

Newton's method is applied to $z^3 - 1$. $f$ has 8 3-cycles. Two of these satisfy $f(z) = e^{\pm 2\pi i/3} z$, or $19z^6 + 7z^3 + 1 = 0$, and the remaining six are obtained from

$$\begin{aligned} h_3(a) &= 256\,a^6 + 1296\,a^3 + 31941 \;=\; 0 \qquad \text{and} \\ j_3(z,\,a) &= (720\,a^3 + 1521)\,z^3 + (-720\,a^4 - 1521\,a)\,z^2 + (224\,a^5 + 1170\,a^2)\,z \\ &+ \;(-168\,a^3 + 3042) \;=\; 0\;. \end{aligned}$$

# References


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## Implementation in Maple

```
> readlib(factors): with(numtheory, mobius): n := 4; # adjust n !!!
> f[0] := z: for i from 1 to n do f[i] := evala(f[i-1]^2 + c) od:
> g := 1: for i from 1 to n do
> if irem(n, i) = 0 then g := g*(f[i] - z)^mobius(n/i) fi od:
> g := sort(evala(g), [z, c], plex); kn := degree(g, z)/n:
> if n <= 3 then Galois_g_z := galois(subs(c = 1, g)) fi;
> q := z - a: for i from 1 to n - 1 do q := q + f[i] od:
> q := collect(q, z): p := g:
> while degree(q, z) > n do
> r0 := collect(evala(Prem(p, q, z)), z): p := q:
> for rf in factors(r0)[2] do if degree(rf[1], z) > 0 then q := rf[1] fi od:
> od:
> r0 := collect(evala(Prem(p, q, z)), z):
> for rf in factors(r0)[2] do
> if degree(rf[1], z) = 0 and degree(collect(rf[1], a), a) = kn
> then h := rf[1] fi od:
> h := collect(h, a): lch := collect(lcoeff(h, a), c):
> if degree(lch, c) = 0 then h := collect(evala(h/lch), a) fi:
> h :=  sort(h, [a, c], plex);
> j := collect(evala(Prem(q, h, a)), z):
> if lcoeff(j, z) = -1 then j := -j fi: j := sort(j, [z, a, c], plex);
> u := mu*lcoeff(j, z) - (-1)^n*tcoeff(j, z):
> r0 := resultant(h, u, a):
> for rf in factors(r0)[2] do
> if degree(rf[1], mu) > 0 then k := rf[1] fi od:
> k := collect(k, c):
> if lcoeff(k, c) = -1 then k := collect(evala(-k), c) fi:
> k := sort(k, [c, mu], plex);
```