Geometrical Approach to Inverse Scattering for the Dirac Equation

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Abstract

The high-energy-limit of the scattering operator for multidimensional relativistic dynamics, including a Dirac particle in an electromagnetic field, is investigated by using time-dependent, geometrical methods. This yields a reconstruction formula, by which the field can be obtained uniquely from scattering data.

I Introduction

For self-adjoint operators H_0 and $H = H_0 + V$ with H_0 having continuous spectrum, the wave operators are defined by $\Omega_\pm = s - \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}$. If they exist on $\mathcal{H}$ and their ranges equal $\mathcal{H}^{ac}(H)$, the scattering system is called complete and the scattering operator $S = \Omega_+^* \Omega_-$ is unitary [11]. The inverse problem is to determine V, given S (and H_0). In [4, 5, 6, 7] Enss and Weder show that for the Schrödinger operator $H_0 = -1/2\,\Delta$ and a translation in momentum space by $\mathbf{v} = v\boldsymbol{\omega}$, $\boldsymbol{\omega} \in S^{\nu-1}$ the high-energy-limit of the scattering operator is given by

$$\Big(\Phi, iv\Big(e^{-i\mathbf{v}\mathbf{x}} S e^{i\mathbf{v}\mathbf{x}} - 1\Big)\Psi\Big) \longrightarrow \int_{-\infty}^{+\infty} d\tau \,\Big(\Phi, V(\mathbf{x} + \boldsymbol{\omega}\tau)\Psi\Big) \quad \text{as } v \to \infty \qquad (1)$$

for suitable Φ, Ψ. The short-range potential V, a multiplication operator, can be uniquely reconstructed from this X-ray transform. This approach generalizes to multiparticle systems and long-range potentials.

Following these ideas, we use time-dependent, geometrical methods to study relativistic quantum mechanics, in particular the Dirac equation with the free Hamiltonian $H_0 = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m$. The main result is Theorem 3.2:

$$s - \lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}_{NW}} S_\pm e^{i\mathbf{v}\mathbf{x}_{NW}} = \exp\{-i\int_{-\infty}^{+\infty} (A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega}t)dt\} \qquad (2)$$

from which the electromagnetic field $(A_0, \mathbf{A})$ may be reconstructed. Here $S_\pm$ describe the scattering of positive/negative energy states in the Foldy-Wouthuysen-representation, and $\mathbf{x}_{NW}$ is the Newton-Wigner position operator. The A_i are supposed to be continuous and to decay integrably, i.e. $\int_0^\infty dR\, \sup_{|\mathbf{x}|\geq R} |\mathbf{A}_i(\mathbf{x})| < \infty$. In [8] Ito has given a similar reconstruction formula for the high-energy-limit of the scattering amplitude using stationary methods, for $A_i \in C^2$ satisfying $|A_i(\mathbf{x})| < c|\mathbf{x}|^{-3-\varepsilon}$.

The charge e is incorporated in A_i, furthermore, we let $c = \hbar = 1$. Note that $-\int_{-\infty}^{+\infty} (A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}+\boldsymbol{\omega}t)dt$ is the classical action of a particle moving along a line with velocity $\boldsymbol{\omega}$, as expected in the semi-classical limit. Introducing suitable units and letting $c \to \infty$ in the r.h.s. of (2) yields $\exp\{i\int_{-\infty}^{+\infty} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x} + \boldsymbol{\omega}t)dt\}$, which has been obtained by Arians [1] as the high-energy-limit of S for a Schrödinger particle in an electromagnetic field.

For mathematical quantum mechanics we refer to [9, 10, 11] and for the Dirac equation to [13]. In Section 2 we study $H_0 = \sqrt{\mathbf{p}^2 + m^2}$, which is similar to the Dirac operator, while being easier to handle. In Section 3 we examine the reconstruction formula for the Dirac equation, which is proved in Section 4. Various generalizations are discussed in Section 5.

II Reconstruction Formula for the Scalar Relativistic Hamiltonian

We consider $\mathcal{H} = L^2(\mathbb{R}^\nu)$ and $H_0 = \sqrt{\mathbf{p}^2 + m^2}$ with $m \geq 0$ and $\mathbf{p} = -i\nabla$. This scalar Hamiltonian H_0 is self-adjoint on the Sobolev-space $H^1(\mathbb{R}^\nu)$ as its domain. It may be considered as a model for relativistic quantum mechanics, since the symbols of the Klein-Gordon- and the Dirac equation have the eigenvalues $\pm\sqrt{\mathbf{p}^2 + m^2}$.

Definition 2.1 (Short-range Potentials) *A symmetric multiplication operator V is called a short-range potential, if it is H_0-bounded with relative bound < 1 and satisfies*

$$\left\| V \sqrt{\mathbf{p}^2 + 1}^{\,-1} F(|\mathbf{x}| > R) \right\| \in L^1\left([0,\infty), dR\right). \qquad (3)$$

$F(\ldots)$ *denotes multiplication with the characteristic function of the indicated region in* $\mathbf{x}$ *-space.*

This definition corresponds to that of the Schrödinger case [4, 5, 6, 7]. Local singularities of V are possible: If, e.g., $\nu > 1$, $p > \nu$ and $V \in L^p + L^\infty$ with $\|V\chi(|\mathbf{x}| > R)\|_{L^p+L^\infty} \in L^1([0,\infty), dR)$, then V is short-range. (The norm is defined by $\|f\|_{L^p+L^\infty} := \inf\{\,\|f_1\|_p + \|f_2\|_\infty | \, f = f_1 + f_2\}$.) For $\nu = 3$, a Yukawa-potential is also admitted, if the coupling constant is small. For a short-range potential V and $H = H_0 + V$ the completeness of the scattering system follows from Theorem 2.1 in [12].

For $\Psi \in \mathcal{H}$, the $\mathbf{x}$-representation is given by $\psi(\mathbf{x})$ and the Fourier transform $\hat\psi(\mathbf{p})$ yields the momentum representation of Ψ. The position operator $\mathbf{x}$ generates translations in momentum space, in particular for any $\mathbf{v} = v\boldsymbol{\omega} \in \mathbb{R}^\nu$

$$e^{-i\mathbf{v}\mathbf{x}} H\, e^{i\mathbf{v}\mathbf{x}} = e^{-i\mathbf{v}\mathbf{x}} \left(\sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{x})\right) e^{i\mathbf{v}\mathbf{x}} = \sqrt{(\mathbf{p}+\mathbf{v})^2 + m^2} + V(\mathbf{x})\,.$$

Lemma 2.2 (Integrable Bound) *Let* V *be a short-range potential. For* Ψ *with* $\hat\psi \in C_0^\infty$ *there are* $v_0 > 0$, $h \in L^1(\mathbb{R})$ *such that*

$$\|V e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}} \Psi\| \le h(t) \quad \textit{for } t \in \mathbb{R}, \textit{ uniformly in } v \ge v_0. \tag{4}$$

This v-independent integrable bound will be crucial to apply the dominated convergence theorem in the proof of Theorem 2.3 .

Proof: We first show that there are $c, v_0 > 0$ such that

$$\left| e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}} \sqrt{\mathbf{p}^2 + 1}\, \Psi \right| (\mathbf{x}) < \frac{c}{(1+|t|)^{\frac{\nu+3}{2}}} \text{ for } |\mathbf{x}| < \frac{|t|}{2}, v > v_0. \tag{5}$$

This follows by a non-stationary phase estimate [13, p.33], [11, p.37] from

$$\left(e^{-i\mathbf{v}\mathbf{x}} e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}} \sqrt{\mathbf{p}^2+1}\, \Psi \right)(\mathbf{x}) = (2\pi)^{-\frac{\nu}{2}} \int dp\, e^{itf(\mathbf{p};\mathbf{x},t,\mathbf{v})} \sqrt{\mathbf{p}^2+1}\, \hat\psi(\mathbf{p}) \tag{6}$$

with $f = \left(\mathbf{p}\cdot\frac{\mathbf{x}}{t} - \sqrt{(\mathbf{p}+\mathbf{v})^2 + m^2}\right)$, since there is a $v_0 > 0$ such that $|\nabla_{\mathbf{p}} f| > 1/4$ for $|\mathbf{x}/t| < 1/2, v > v_0$, $\mathbf{p} \in \operatorname{supp}(\hat\psi)$. Also $\partial_{\mathbf{p}}^\beta f$ is bounded there for $|\beta| > 0$, and $\partial_{\mathbf{p}}^\gamma \left(\sqrt{\mathbf{p}^2+1}\hat\psi(\mathbf{p})\right)$ is bounded for $|\gamma| \ge 0$. Now we consider

$$\begin{aligned}
& \left\| e^{-i\mathbf{v}\mathbf{x}} V e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}}\, \Psi \right\| \\
= & \left\| V \sqrt{\mathbf{p}^2+1}^{\,-1} \left\{ F\Big(|\mathbf{x}| > \frac{|t|}{2}\Big) + F\Big(|\mathbf{x}| < \frac{|t|}{2}\Big) \right\} e^{-it\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}} \sqrt{\mathbf{p}^2+1}\, \Psi \right\| \\
\le & \left\| V \sqrt{\mathbf{p}^2+1}^{\,-1} F\Big(|\mathbf{x}| > \frac{|t|}{2}\Big) \right\| \cdot \left\| \sqrt{\mathbf{p}^2+1}\, \Psi \right\| \\
+ & \left\| V \sqrt{\mathbf{p}^2+1}^{\,-1} \right\| \cdot \left\| F\Big(|\mathbf{x}| < \frac{|t|}{2}\Big) e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}} \sqrt{\mathbf{p}^2+1} \Psi \right\|
\end{aligned}$$

The first term is in L^1 by (3) and the second is bounded by $\tilde{c}/(1+|t|)^{\frac{3}{2}}$ by (5). ■

This decomposition is motivated by the geometrical idea that the wave-packet is concentrated around $\boldsymbol{\omega}t$, where the potential is small, and that it is small around the origin, where the potential may be large.

Theorem 2.3 (High-Energy-Asymptotics of S) *For a short-range potential V the high-energy limit of the scattering operator is given by*

$$s-\lim_{v\to\infty} e^{-i\mathbf{vx}} S e^{i\mathbf{vx}} = \exp\{-i\int_{-\infty}^{+\infty} V(\mathbf{x}+\boldsymbol{\omega}t)dt\}, \tag{7}$$

the integral being convergent for a.e. $\mathbf{x}\in\mathbb{R}^\nu$.

Proof: It is sufficient to consider the dense set of Ψ with $\hat{\psi}\in C_0^\infty$.

$$\begin{aligned} e^{-i\mathbf{vx}}\Omega_+ e^{i\mathbf{vx}}\,\Psi &= \Psi + \int_0^\infty dt\,\frac{d}{dt} e^{-i\mathbf{vx}} e^{iHt} e^{-iH_0 t} e^{i\mathbf{vx}}\,\Psi \\ &= \Psi + i\int_0^\infty dt\, e^{i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}+V(\mathbf{x})\right)t} V(\mathbf{x}) e^{-i\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}\,t}\Psi \\ &= \Psi + i\int_0^\infty dt\, e^{i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}-v+V(\mathbf{x})\right)t} V(\mathbf{x}) e^{-i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}-v\right)t}\Psi. \end{aligned}$$

With $\mathbf{v}=v\boldsymbol{\omega}$ we find $(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}-v+V)\overset{v\to\infty}{\longrightarrow}\boldsymbol{\omega}\cdot\mathbf{p}+V$ in the strong resolvent sense and the exponential converges strongly [9, Theorems VIII.25,21]. $V\sqrt{\mathbf{p}^2+1}^{\,-1}$ is bounded and

$$e^{-i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}-v\right)t}\sqrt{\mathbf{p}^2+1}\,\Psi \overset{v\to\infty}{\longrightarrow} e^{-i\boldsymbol{\omega}\mathbf{p}t}\sqrt{\mathbf{p}^2+1}\,\Psi.$$

The integrand is bounded by $h(t)$ independent of $v>v_0$ by Lemma 2.2 . Using the dominated convergence theorem (for the Bochner-integral [3]) we conclude that

$$\begin{aligned} \lim_{v\to\infty} e^{-i\mathbf{vx}}\Omega_+ e^{i\mathbf{vx}}\Psi &= \Psi + i\int_0^\infty dt\, e^{i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)t} V(\mathbf{x}) e^{-i\boldsymbol{\omega}\mathbf{p}t}\Psi \\ &= \lim_{s\to\infty} e^{i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)s} e^{-i\boldsymbol{\omega}\mathbf{p}s}\Psi. \end{aligned} \tag{8}$$

In the special case of continuous V with integrable decay , (8) is shown to equal $\exp\{i\int_0^\infty dt\, V(\mathbf{x}+\boldsymbol{\omega}t)\}\Psi$ by considering the family of unitary operators $U(s) = e^{i\boldsymbol{\omega}\mathbf{p}s} e^{-i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)s}$, which satisfies the differential equation

$$i\dot{U}(s) = e^{i\boldsymbol{\omega}\mathbf{p}s} V(\mathbf{x})\, e^{-i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)s} = V(\mathbf{x}+\boldsymbol{\omega}s)U(s)$$

and $U(0)=1$, as does $\exp\{-i\int_0^s dt\, V(\mathbf{x}+\boldsymbol{\omega}t)\}$. For general V we use the decomposition $V=V_+-V_-,\ V_\pm\geq 0$ and choose $V_{n,\pm}\in C_0^0$ with $V_{n,\pm}\nearrow V_\pm$ a.e.. Let

$V_n := V_{n,+} - V_{n,-}$. The estimate of the integral by an integrable function h from Lemma 2.2 also holds for $V_\pm, V_{n,\pm}, V_n$, uniformly in n. Again, by applying the dominated convergence theorem once more, we deduce

$$\begin{aligned}\lim_{v\to\infty} e^{-i\mathbf{vx}}\Omega_+ e^{i\mathbf{vx}}\Psi &= \Psi + i\lim_{n\to\infty}\int_0^\infty dt\, e^{i(\boldsymbol{\omega}\mathbf{p}+V_n)t}V_n e^{-i\boldsymbol{\omega}\mathbf{p}t}\Psi \\ &= \lim_{n\to\infty}\exp\{i\int_0^\infty dt\, V_n(\mathbf{x}+\boldsymbol{\omega}t)\}\,\Psi.\end{aligned}$$

Also we see that for a subsequence $\lim_{n\to\infty}\exp\{i\int_0^\infty dt\, V'_{n,\pm}(\mathbf{x}+\boldsymbol{\omega}t)\}$ exists for a.e. $\mathbf{x}\in\mathbb{R}^\nu$, therefore $\int_0^\infty dt\, V_\pm(\mathbf{x}+\boldsymbol{\omega}t)$ exists for a.e. $\mathbf{x}$ since for a.e. $\mathbf{x}$ the monotone convergence $V'_{n,\pm}(\mathbf{x}+\boldsymbol{\omega}t)\nearrow V_\pm(\mathbf{x}+\boldsymbol{\omega}t)$ holds for a.e. $t\in\mathbb{R}$. We consider Ω_- similarly and find

$$s-\lim_{v\to\infty} e^{-i\mathbf{vx}}\Omega_\pm e^{i\mathbf{vx}} = \exp\{i\int_0^{\pm\infty} dt\, V(\mathbf{x}+\boldsymbol{\omega}t)\}. \tag{9}$$

Using $S=\Omega_+^*\Omega_-$ we get the desired reconstruction formula

$$w-\lim_{v\to\infty} e^{-i\mathbf{vx}}Se^{i\mathbf{vx}} = \exp\{-i\int_{-\infty}^{+\infty} V(\mathbf{x}+\boldsymbol{\omega}t)dt\}. \tag{10}$$

The unitarity of $e^{-i\mathbf{vx}}Se^{i\mathbf{vx}}$ and of its weak limit imply the strong convergence. ■

The proof suggests the following physical interpretation: If the velocity of a particle approaches 1 (i.e. the speed of light), the spreading of the wave-packet is negligible and the free kinematics reduce to a pure translation.

Theorem 2.4 (Injectivity of the Scattering Map) *Consider* $\nu > 1$ *and* $\mathcal{V} := \{V\in C^0(\mathbb{R}^\nu,\mathbb{R})\,|\, \|V\chi(|\mathbf{x}|>R)\|_\infty \in L^1([0,\infty),dR)\}$. *Then the scattering map:*

$$\begin{aligned}\mathcal{V} &\to \mathcal{L}(\mathcal{H})\\ V &\mapsto S=S(H_0,H_0+V)\end{aligned}$$

is injective, i.e. S determines V uniquely.

The case $\nu=1$ cannot be treated with our methods. The case of more general V shall be a topic of further research, the difficulties arising from the non-injectivity of the exponential function.

Proof: Theorem 2.3 yields $\exp\{-i\int_{-\infty}^{+\infty} V(\mathbf{x}+\boldsymbol{\omega}t)dt\}$ as a continuous function of $\mathbf{x}$, thus giving $\int_{-\infty}^{+\infty} V(\mathbf{x}+\boldsymbol{\omega}t)dt$ up to a fixed multiple of 2π. This X-ray transform is obtained uniquely, since it must vanish as $|\mathbf{x}|\to\infty$ orthogonal to $\boldsymbol{\omega}$. If $\int_{-\infty}^{+\infty}dt\, V(\mathbf{x}+\boldsymbol{\omega}t)=0$, then considering $\boldsymbol{\omega}t$ as polar coordinates for $\mathbb{R}^\nu$ yields

$$0=\int_{S^{\nu-1}}d\boldsymbol{\omega}\int_{-\infty}^{+\infty}dt\, V(\mathbf{x}+\boldsymbol{\omega}t) = 2\int d\mathbf{y}\frac{V(\mathbf{x}+\mathbf{y})}{|\mathbf{y}|^{\nu-1}}. \tag{11}$$

To show that V is determined uniquely, we examine the case $\nu = 2$ first. Here $V \in L^2$ and $\frac{1}{|\mathbf{p}|^{1-\varepsilon}} \in L^2_{loc}$ imply $\frac{1}{|\mathbf{p}|^{1-\varepsilon}}\hat{V} \in L^1_{loc}$. As a tempered distribution

$$0 = \mathcal{S}' - \lim_{\varepsilon \to 0} \frac{2}{|\mathbf{x}|^{1+\varepsilon}} * V\,. \tag{12}$$

We conclude that $\mathcal{S}' - \lim_{\varepsilon\to 0} \frac{2}{|\mathbf{p}|^{1-\varepsilon}}\hat{V} = 0$, therefore $\hat{V}(\mathbf{p}) = 0$ a.e. . For $\nu > 2$, this argument shows that the restriction of V to any 2-plane in $\mathbb{R}^\nu$ is determined uniquely. ∎

III Reconstruction Formula for the Dirac Equation

Let $\mathcal{H} := L^2(\mathbb{R}^\nu, \mathbb{C}^\mu)$ and $H_0 := \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m$ with $m \geq 0$ and anticommuting, symmetric, unitary matrices $\alpha_1, \ldots, \alpha_\nu, \beta$. The most interesting case is $\nu = 3$, $\mu = 4$. H_0 is self-adjoint on $H^1(\mathbb{R}^\nu)$. The symbol of H_0 has the eigenvalues $\pm E$, with the abbreviation $E = +\sqrt{\mathbf{p}^2 + m^2}$. V shall be a symmetric-matrix-valued function. Under conditions analogous to Def.2.1 the scattering system is complete [12, 13], but we need more restrictive conditions to prove the following theorem. An electromagnetic field is described by $V = A_0 - \boldsymbol{\alpha}\cdot\mathbf{A}$, where $-\operatorname{grad} A_0$ is the electric and $\operatorname{rot}\mathbf{A}$ the magnetic field. For $\boldsymbol{\omega} \in S^{\nu-1}$ we make the decomposition $V = V_{+,\boldsymbol{\omega}} + V_{-,\boldsymbol{\omega}}$ with $V_{\pm,\boldsymbol{\omega}} := 1/2(V \pm \boldsymbol{\alpha}\cdot\boldsymbol{\omega} V \boldsymbol{\alpha}\cdot\boldsymbol{\omega})$, which yields $[V_{+,\boldsymbol{\omega}}, \boldsymbol{\alpha}\cdot\boldsymbol{\omega}] = 0$ and $\{V_{-,\boldsymbol{\omega}}, \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\} = 0$. For the e.m. field we get $V_{+,\boldsymbol{\omega}} = A_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,\boldsymbol{\omega}\cdot\mathbf{A}$. As operators, the A_i are functions of the standard position operator $\mathbf{x}$, which generates momentum translations in the standard representation. We will discuss the alternative Newton-Wigner position operator $\mathbf{x}_{NW}$ below.

Theorem 3.1 (High-Energy-Asymptotics of S for the Dirac Equation) *Suppose the components of the symmetric-matrix-valued multiplication operator V are continuous with integrable decay, i.e. $\|VF(|\mathbf{x}| > R)\| \in L^1([0,\infty), dR)$, and the matrices $V_{+,\boldsymbol{\omega}}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^\nu$ commute, i.e. $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1), V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)] = 0$ for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^\nu$. Then*

$$s - \lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}} S e^{i\mathbf{v}\mathbf{x}} = \exp\{-i\int_{-\infty}^{+\infty} V_{+,\boldsymbol{\omega}}(\mathbf{x} + \boldsymbol{\omega}t)dt\}. \tag{13}$$

For an electromagnetic field this is $\exp\{-i\int_{-\infty}^{+\infty}(A_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,\boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x} + \boldsymbol{\omega}t)dt\}$.

If the condition $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1), V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)] = 0$ is violated, the exponential must be replaced by a time-ordered product. Theorem 3.1 will be proved in the next section, after having examined its consequences. Theorem 3.2 gives a modified reconstruction formula, adapted to positive energy states. Theorem 3.3 shows how to obtain the electromagnetic field from scattering data. A general matrix-valued potential cannot be recovered from (13), since for $V = \beta\Phi$, Φ real-valued we have $V_{+,\boldsymbol{\omega}} = 0$.

For $m > 0$ and $\beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ consider the Foldy-Wouthuysen-transform $U(\mathbf{p}) = \sqrt{\frac{E+m}{2E}}\Big(\mathbf{1} + \beta\frac{\boldsymbol{\alpha}\mathbf{p}}{E+m}\Big)$, which diagonalizes H_0: $U(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)U^{-1} = E\beta$. The Foldy-Wouthuysen-representation of Ψ is given by $\hat{\psi}_{FW}(\mathbf{p}) = U(\mathbf{p})\hat{\psi}(\mathbf{p})$. S is decomposed to $S_{FW} = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}$, where $S_\pm$ are unitary operators on $L^2(\mathbb{R}^\nu, \mathbb{C}^{\mu/2})$ and describe the scattering of electrons with positive/negative energy. The Newton-Wigner position operator $\mathbf{x}_{NW}$ is the generator of momentum translations in the FW-representation and acts on ψ_{FW} as multiplication with the coordinate function. It is given by $U^*(\mathbf{p})\mathbf{x}U(\mathbf{p})$ in the standard representation. In contrast to $\mathbf{x}$, the operator $\mathbf{x}_{NW}$ does not mix the states with positive/negative energy. This suggests to investigate $e^{-i\mathbf{v}\mathbf{x}_{NW}}S\,e^{i\mathbf{v}\mathbf{x}_{NW}}$, which is decomposed to $e^{-i\mathbf{v}\mathbf{x}_{NW}}S_\pm e^{i\mathbf{v}\mathbf{x}_{NW}}$, where the restriction of the Newton-Wigner position operator to the positive/negative energy subspaces is also denoted by $\mathbf{x}_{NW}$.

Theorem 3.2 (High-Energy-Asymptotics of $S_\pm$) *Suppose that A_0, $\mathbf{A}$ are continuous with integrable decay. Then*

$$s-\lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}_{NW}}S_\pm e^{i\mathbf{v}\mathbf{x}_{NW}} = \exp\{-i\int_{-\infty}^{+\infty}(A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega}t)dt\} \qquad (14)$$

Thus the limit of S_+ acts on positive energy states independent of spin.
Proof: In the standard representation, Theorem 3.1 yields

$$\begin{aligned}
& e^{-i\mathbf{v}\mathbf{x}_{NW}}S\,e^{i\mathbf{v}\mathbf{x}_{NW}} \\
= \;& U^*(\mathbf{p})\,e^{-i\mathbf{v}\mathbf{x}}U(\mathbf{p})\,S\,U^*(\mathbf{p})\,e^{i\mathbf{v}\mathbf{x}}U(\mathbf{p}) \\
= \;& U^*(\mathbf{p})\,U(\mathbf{p}+\mathbf{v})\,e^{-i\mathbf{v}\mathbf{x}}S\,e^{i\mathbf{v}\mathbf{x}}U^*(\mathbf{p}+\mathbf{v})\,U(\mathbf{p}) \\
\stackrel{v\to\infty}{\longrightarrow} \;& U^*(\mathbf{p})\frac{1}{\sqrt{2}}(1 + \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\,e^{-i\int_{-\infty}^{+\infty}\big(A_0 - \boldsymbol{\alpha\omega}\,\boldsymbol{\omega}\mathbf{A}\big)(\mathbf{x}+\boldsymbol{\omega}t)dt}\frac{1}{\sqrt{2}}(1 - \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega})U(\mathbf{p}) \\
= \;& U^*(\mathbf{p})\,e^{-i\int_{-\infty}^{+\infty}\big(A_0 - \beta\,\boldsymbol{\omega}\mathbf{A}\big)(i\nabla_{\mathbf{p}}+\boldsymbol{\omega}t)dt}U(\mathbf{p})\,,
\end{aligned}$$

where we used $\frac{1}{\sqrt{2}}(1 + \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\,\boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,\frac{1}{\sqrt{2}}(1 - \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega}) = \beta$. By changing to the FW-representation, the above expression becomes

$$\exp\{-i\int_{-\infty}^{+\infty}(A_0 - \beta\,\boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega}t)dt\}\ .$$

The block-structure of β yields the desired result. ∎

Theorem 3.3 (Injectivity of the Scattering Map) *Consider* $\nu > 1$ *and* $\mathcal{V} := \{(A_0, \mathbf{A}) \,|\, A_i \in C^0(\mathbb{R}^\nu, \mathbb{R}), \|A_i \chi(|\mathbf{x}| > R)\|_\infty \in L^1([0,\infty), dR), \; \mathbf{A} \in L^2(\mathbb{R}^\nu, \mathbb{R}^\nu)\}$. *Then the scattering map:*

$$(A_0, \mathbf{A}) \mapsto S_+$$

is injective on $\mathcal{V}$ *except for gauge-invariance, i.e.* S_+ *determines* A_0 *uniquely and* $\mathbf{A}$ *up to a gradient. Thus* $\mathbf{E} = -gradA_0$, $\mathbf{B} = rot\mathbf{A}$ *are determined uniquely.*

$\mathbf{A}$ cannot be determined uniquely, for if λ is vanishing at ∞, then S remains unchanged when $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\lambda$. This corresponds to the facts that the phase of a wave-function at a single point has no physical meaning, and that only rot$\mathbf{A}$ is measurable.

Proof: With Theorem 3.2 we get $a(\mathbf{x}, \boldsymbol{\omega}) := \int_{-\infty}^{+\infty} (A_0 - \boldsymbol{\omega} \cdot \mathbf{A})(\mathbf{x} + \boldsymbol{\omega} t) dt$ as in Theorem 2.4. Now $1/2\,(a(\mathbf{x}, \boldsymbol{\omega}) + a(\mathbf{x}, -\boldsymbol{\omega})) = \int_{-\infty}^{+\infty} A_0(\mathbf{x} + \boldsymbol{\omega} t) dt$, which determines A_0 uniquely, and$1/2\,(a(\mathbf{x}, -\boldsymbol{\omega}) - a(\mathbf{x}, \boldsymbol{\omega})) = \int_{-\infty}^{+\infty} \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t) dt$, which determines $\mathbf{A}$ up to a gradient (Lemma 3.4). ∎

To prove that lemma we need the extra assumption $\mathbf{A} \in L^2$. In [8, Lemma 3.4] a different way to reconstruct $\mathbf{B} = \mathrm{rot}\,\mathbf{A}$ is proposed, which does not need $\mathbf{A} \in L^2$ but $\mathbf{A} \in C^1$ with $\mathbf{B}$ decaying integrably.

Lemma 3.4 (Reconstruction of A) *For* $\nu > 1$, *consider* $\mathbf{A} \in C^0 \cap L^2(\mathbb{R}^\nu, \mathbb{R}^\nu)$ *having integrable decay. Then* $\mathbf{A}$ *is determined up to* $\nabla\lambda$ *by* $\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t)$.

Proof: As in (11) one has $\int_{S^{\nu-1}} d\boldsymbol{\omega}\, \boldsymbol{\omega} \int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega} \cdot \mathbf{A}(\cdot + \boldsymbol{\omega} t) = 2 \frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^{\nu+1}} * \mathbf{A}$,which is a bounded, continuous function of $\mathbf{x}$. We will show that $\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t) = 0$ implies $\mathbf{A} = \nabla\lambda$ for a λ vanishing at ∞. In the case $\nu = 3$ one finds $(2 \frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^4})^\wedge = \sqrt{\frac{\pi}{2}} \frac{1}{|\mathbf{p}|} (1 - \frac{\mathbf{p}\mathbf{p}^T}{|\mathbf{p}|^2}) \in L^2 + L^\infty$ and $\sqrt{\frac{\pi}{2}} \frac{1}{|\mathbf{p}|} (1 - \frac{\mathbf{p}\mathbf{p}^T}{|\mathbf{p}|^2}) \hat{\mathbf{A}} \in L^1 + L^2$, thus

$$2 \frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^4} * \mathbf{A} = 2\pi^2 (\frac{1}{|\mathbf{p}|} \hat{\mathbf{A}} - \frac{\mathbf{p}}{|\mathbf{p}|^3} \mathbf{p} \cdot \hat{\mathbf{A}})^\vee \in L^2 + L^\infty. \tag{15}$$

Now $\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t) = 0$ implies $\hat{\mathbf{A}} = \frac{\mathbf{p}}{|\mathbf{p}|^2} \mathbf{p} \cdot \hat{\mathbf{A}}$ a.e. , thus $\mathbf{A} = \nabla\lambda$ for a $\lambda \in L^6_w$. For $\nu > 3$, the proof is similar, but for $\nu = 2$ it must be modified to include a $\mathcal{S}'$-limit as in (12). ∎

Remark: In the Coulomb-gauge div$\mathbf{A} = 0$ (in the sense of $\mathcal{S}'$), $\mathbf{A}$ is determined uniquely by the integral transform: For $\nu = 2$, one has

$$\int_{-\infty}^{+\infty} dt\, \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t) = \boldsymbol{\omega} \int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t),$$

and the proof of theorem 2.4 applies to the components A_1, A_2. For $\nu = 3$, (15) implies $\int_{S^2} d\boldsymbol{\omega}\, \boldsymbol{\omega} \int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t) = 2\pi^2 (\frac{1}{|\mathbf{p}|} \hat{\mathbf{A}})^\vee$. For $\nu > 3$, the factor $2\pi^2$ must be replaced by $2\pi^{\frac{\nu+1}{2}} / \Gamma(\frac{\nu+1}{2})$.

IV Proof of Theorem 3.1

We first discuss two preparatory lemmata:

Lemma 4.1 (Approximation of S) *Let V be a matrix-valued multiplication operator, which is H_0-bounded with relative bound < 1 and satisfies*

$$\|V\sqrt{\mathbf{p}^2+1}^{\,-1}F(|\mathbf{x}|>R)\| \in L^1\left([0,\infty),dR\right).$$

Then for Φ,Ψ with $\hat{\phi},\hat{\psi}\in C_0^\infty$ there is $v_0>0$ such that

$$\lim_{t\to\infty}\Big(\Phi, e^{-i\mathbf{v}\mathbf{x}}e^{iH_0t}e^{-i2Ht}e^{iH_0t}e^{i\mathbf{v}\mathbf{x}}\Psi\Big) = \Big(\Phi, e^{-i\mathbf{v}\mathbf{x}}S\,e^{i\mathbf{v}\mathbf{x}}\Psi\Big) \tag{16}$$

uniformly in $v>v_0$.

Proof: We first note that for $\hat{\phi}\in C_0^\infty$ there are $c,v_0>0$ such that

$$\left|e^{-iH_0t}e^{i\mathbf{v}\mathbf{x}}\sqrt{\mathbf{p}^2+1}\,\Phi\right|(\mathbf{x}) < \frac{c}{(1+|t|)^{\frac{\nu+3}{2}}} \text{ for } |\mathbf{x}|<\frac{|t|}{2}, v>v_0. \tag{17}$$

This is shown as in Lemma 2.2 by observing that

$$e^{-i\mathbf{v}\mathbf{x}}e^{-iH_0t}e^{i\mathbf{v}\mathbf{x}} = e^{-i\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}\,t}P_{+,\mathbf{v}} + e^{i\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}\,t}P_{-,\mathbf{v}} \tag{18}$$

with $P_{\pm,\mathbf{v}} = \frac{1}{2}\Big(1\pm\frac{\boldsymbol{\alpha}(\mathbf{p}+\mathbf{v})+\beta m}{\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}}\Big)$ and that $\partial_{\mathbf{p}}^{\gamma}\left(P_{\pm,\mathbf{v}}\sqrt{\mathbf{p}^2+1}\hat{\phi}(\mathbf{p})\right)$ is bounded for $|\gamma|\geq 0$. Using the same decomposition as in the proof of Lemma 2.2, we find $h\in L^1$ such that $\|Ve^{-iH_0t}e^{i\mathbf{v}\mathbf{x}}\Phi\|\leq h(t)$ for $t\in\mathbb{R}$, $v\geq v_0$. Now

$$\begin{aligned}\|\Omega_+e^{i\mathbf{v}\mathbf{x}}\Phi - e^{iHt}e^{-iH_0t}e^{i\mathbf{v}\mathbf{x}}\Phi\| &= \|i\int_t^\infty ds\,e^{iHs}Ve^{-iH_0s}e^{i\mathbf{v}\mathbf{x}}\Phi\| \\ &\leq \int_t^\infty ds\,h(s)\to 0 \quad \text{as}\, t\to\infty\end{aligned}$$

uniformly in $v>v_0$. We treat Ω_- and Ψ analogously, and the result for S is obtained. ∎

Lemma 4.2 (Limit for finite t) *Suppose V has bounded, continuous components and satisfies $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1),V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)]=0$. Then for all $t>0$*

$$s-\lim_{v\to\infty}e^{-i\mathbf{v}\mathbf{x}}e^{iH_0t}e^{-i2Ht}e^{iH_0t}e^{i\mathbf{v}\mathbf{x}} = \exp\{-i\int_{-t}^{t}ds\,V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)\}. \tag{19}$$

Proof: According to Theorem X.69 in [10], we have the Dyson-expansion

$$e^{iH_0t}e^{-i2Ht}e^{iH_0t}\Psi = \sum_{n=0}^{\infty}(-i)^n \int\limits_{-t<t_1<\ldots<t_n<t} dt_n\ldots dt_1 V(t_n)\ldots V(t_1)\Psi \quad (t>0, \Psi\in\mathcal{H})$$

with $V(s) = e^{iH_0s}V\,e^{-iH_0s}$. In the momentum representation it is easily shown that $s-\lim_{v\to\infty}\left(e^{-i\mathbf{v}\mathbf{x}}e^{-iH_0s}e^{i\mathbf{v}\mathbf{x}} - e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}\right) = 0$, thus

$$\lim_{v\to\infty}\int_{-t}^{t} ds\left(e^{-i\mathbf{v}\mathbf{x}}V(s)e^{i\mathbf{v}\mathbf{x}} - e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V\,e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}\right)\Psi = 0.$$

From $\boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,V_{\pm,\boldsymbol{\omega}} = \pm V_{\pm,\boldsymbol{\omega}}\,\boldsymbol{\alpha}\cdot\boldsymbol{\omega}$ we conclude

$$\begin{aligned} & e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V\,e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s} \\ = \;& e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V_{+,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s} + e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V_{-,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s} \\ = \;& e^{i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}V_{+,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega\omega}\mathbf{p}s} + e^{i2\boldsymbol{\alpha\omega}vs}e^{i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}V_{-,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}. \end{aligned}$$

The Riemann-Lebesgue Lemma yields

$$\lim_{v\to\infty}\int_{-t}^{t} ds\,e^{i2\boldsymbol{\alpha\omega}vs}e^{i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}V_{-,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}\Psi = 0$$

and thus $\lim_{v\to\infty}\int_{-t}^{t}ds\,e^{-i\mathbf{v}\mathbf{x}}V(s)e^{i\mathbf{v}\mathbf{x}}\Psi = \int_{-t}^{t}ds\,W(s)\Psi$ with $W(s) = e^{i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}V_{+,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}$. By induction, it is shown that

$$\begin{aligned} \lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}} \int\limits_{-t<t_1<\ldots<t_n<t} dt_n\ldots dt_1 V(t_n)\ldots V(t_1)e^{i\mathbf{v}\mathbf{x}}\Psi &= \\ \int\limits_{-t<t_1<\ldots<t_n<t} dt_n\ldots dt_1 W(t_n)\ldots W(t_1)\Psi &= \\ \frac{1}{n!}\int\limits_{[-t,t]^n} dt_n\ldots dt_1 W(t_n)\ldots W(t_1)\Psi &= \frac{1}{n!}\left(\int\limits_{-t}^{t} ds\,W(s)\right)^n\Psi. \end{aligned}$$

Here the time-ordering in the integral was resolved because $W(s)$ is a family of multiplication operators satisfying $[W(s_1), W(s_2)] = 0$ for $s_1,\, s_2\in\mathbb{R}$, which follows from $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1), V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)] = 0$ and $[\boldsymbol{\alpha}\cdot\boldsymbol{\omega}, V_{+,\boldsymbol{\omega}}(\mathbf{x})] = 0$, observing that $W(s) = 1/2(1+\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\,V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s) + 1/2(1-\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\,V_{+,\boldsymbol{\omega}}(\mathbf{x}-\boldsymbol{\omega}s)$. This decomposition also yields $\int_{-t}^{t}ds\,W(s) = \int_{-t}^{t}ds\,V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)$. The Dyson-series converges uniformly in v, thus $\lim_{v\to\infty}$ and $\sum_{n=0}^{\infty}$ may be interchanged. ∎

Proof of Theorem 3.1: The hypotheses of Lemmata 4.1, 4.2 are fulfilled. For $\hat{\phi}, \hat{\psi}\in C_0^\infty$ an $\varepsilon/3$-trick shows: The uniform convergence in (16) entails that the following limits may be interchanged.

$$\begin{aligned}
\lim_{v\to\infty}\left(\Phi, e^{-i\mathbf{v}\mathbf{x}}S\,e^{i\mathbf{v}\mathbf{x}}\Psi\right) &= \lim_{v\to\infty}\lim_{t\to\infty}\left(\Phi, e^{-i\mathbf{v}\mathbf{x}}e^{iH_0t}e^{-i2Ht}e^{iH_0t}e^{i\mathbf{v}\mathbf{x}}\Psi\right)\\
&= \lim_{t\to\infty}\lim_{v\to\infty}\left(\Phi, e^{-i\mathbf{v}\mathbf{x}}e^{iH_0t}e^{-i2Ht}e^{iH_0t}e^{i\mathbf{v}\mathbf{x}}\Psi\right)\\
&= \lim_{t\to\infty}\left(\Phi, \exp\{-i\int_{-t}^{t}ds\,V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)\}\,\Psi\right)\\
&= \left(\Phi, \exp\{-i\int_{-\infty}^{\infty}ds\,V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)\}\,\Psi\right)
\end{aligned}$$

A density argument yields weak convergence, and the unitarity of $e^{-i\mathbf{v}\mathbf{x}}Se^{i\mathbf{v}\mathbf{x}}$ and of its weak limit imply the strong convergence. ■

V Summary and Generalizations

For $H_0 = \sqrt{\mathbf{p}^2 + m^2}$ we obtained $\exp\{-i\int_{-\infty}^{+\infty} V(\mathbf{x}+\boldsymbol{\omega}t)dt\}$ from S for very general short-range V, but the reconstruction of V was only accomplished for continuous V with integrable decay.

For $H_0 = \boldsymbol{\alpha}\cdot\mathbf{p}+\beta m$ we obtained $\exp\{-i\int_{-\infty}^{+\infty}(A_0-\boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW}+\boldsymbol{\omega}t)dt\}$ from S_+ for continuous A_i with integrable decay. The proofs of Theorems 3.1, 3.2 extend to the case of $V \in L^\infty$ with $\|V\sqrt{\mathbf{p}^2+1}^{\,-1}F(|\mathbf{x}| > R)\| \in L^1([0,\infty), dR)$. We expect these Theorems to be true for general short-range A_0, but this is not yet proved. Lemma 4.1 holds under very general conditions, but the Dyson-expansion in 4.2 demands that V should be bounded.

The Aharanov-Bohm-experiment suggests to consider the case $\nu = 2$ with the magnetic field $B = \mathrm{rot}\mathbf{A} \in C_0^0$. This requires the following modifications:

- Given B with $\int B \neq 0$, there is no $\mathbf{A}$ of integrable decay, but there are vectorpotentials with $|\mathbf{A}(\mathbf{x})| < c/|\mathbf{x}|$ and $\frac{\mathbf{x}}{|\mathbf{x}|}\cdot\mathbf{A}(\mathbf{x})$ decaying integrably. If $\tilde{\mathbf{A}} = \mathbf{A} + \nabla\lambda$, then $\Lambda(\mathbf{x}) = \lim_{r\to\infty}\lambda(r\mathbf{x})$ exists and may be $\neq 0$.
- Choosing a special gauge with $\mathrm{supp}(\mathbf{A})$ in a cone as in [2], Theorems 3.1, 3.2 are shown to remain valid for this $\mathbf{A}$.
- In a different gauge $\tilde{\mathbf{A}}$ we find $\tilde{S}_\pm = e^{i\Lambda(\pm\mathbf{p})}S_\pm e^{-i\Lambda(\mp\mathbf{p})}$ and conclude that also $e^{-i\mathbf{v}\mathbf{x}_{NW}}\tilde{S}_\pm e^{i\mathbf{v}\mathbf{x}_{NW}} \to \exp\{-i\int_{-\infty}^{+\infty}(A_0 \mp \boldsymbol{\omega}\cdot\tilde{\mathbf{A}})(\mathbf{x}_{NW}+\boldsymbol{\omega}t)dt\}$.
- The gauge-invariance of S is lost, but we expect all physically measurable quantities to be gauge-independent. Under the idealized assumption that phase-differences are measurable in interference-experiments, the high-energy-limit of S_+ yields $a(\mathbf{x},\boldsymbol{\omega}) = \int_{-\infty}^{+\infty}(A_0 - \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}+\boldsymbol{\omega}t)dt$ up to an

additive constant (depending on $\boldsymbol{\omega}$). But A_0 is supposed to decay integrably, thus the symmetric part of $a(\mathbf{x},\boldsymbol{\omega})$ is determined uniquely, from which A_0 is obtained.

- Lemma 3.4 is not applicable, but at least $B \in C_0^1$ may be obtained from the following formula involving differentiation in the direction of $\boldsymbol{\varpi}$ orthogonal to $\boldsymbol{\omega}$:

$$\int_{-\infty}^{+\infty} dt\, B(\boldsymbol{\varpi} s + \boldsymbol{\omega} t) = \frac{d}{ds}\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega}\cdot\mathbf{A}(\boldsymbol{\varpi} s + \boldsymbol{\omega} t) \quad \text{for} \quad \boldsymbol{\varpi} = (\boldsymbol{\omega}_2, -\boldsymbol{\omega}_1)^T. \tag{20}$$

Finally we mention that the Klein-Gordon equation for a charged spin-0-particle can be treated in the same way as the Dirac equation, since the Dyson-expansion also applies to the 2-Hilbertspace-formalism. We find the same result

$$e^{-i\mathbf{v}\mathbf{x}_{NW}} S_\pm e^{i\mathbf{v}\mathbf{x}_{NW}} \;\to\; \exp\{-i\int_{-\infty}^{+\infty} (A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega} t)dt\}. \tag{21}$$

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