Quadratic captures and matings

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Abstract

There is an alternative construction of mating, when at least one polynomial is preperiodic: shift the infinite critical value of the other polynomial to a preperiodic point.

1 Introduction

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2 Background

2.1 Polynomials and rational maps

2.2 The Thurston Theorem

2.3 A path in moduli space

The pullback of homeomorphisms ψ_n was easy to define, but it is not computed easily: repeated pullbacks would be defined piecewise, and solving the Beltrami equation numerically would be impractical as well. The isotopy classes in Teichmüller space are meant to represent only combinatorial information anyway: we are interested in the pullback of marked points $x_i(n) \in \pi(\sigma_g^n([\psi_0]))$ and maps f_n, and the combinatorial description is needed to make a finite choice between different possible preimages. This characterization of the topology has been implemented in terms of spiders [**?**, **?**], medusas [**?**], and triangulations [**?**]. These contain the necessary information from Teichmüller space without using actual homeomorphisms ψ_n.

Following Bartholdi–Nekrashevych [**?**] and Buff–Chéritat [**?**], the following alternative method shall be discussed. It means that Teichmüller space is used explicitly only to check a suitable initialization of a path in moduli space. Afterward the path is pulled back simply by choosing preimages from continuity. The application to matings is discussed in Sections 5 and 6. The spider algorithm is implemented with a path in [?] and further applications to quadratic polynomials are given; twisted polynomials and Lattès maps are discussed in [?] as well.

Proposition 2.1 (Path in moduli space)
Suppose g is a Thurston map of degree $d \geq 2$, and there is a continuous path of homeomorphisms $\psi_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $0 \leq t \leq 1$, with $\psi_0 \circ g = f_0 \circ \psi_1$ for a rational map f_0. So $[\psi_1] = \sigma_g([\psi_0])$.
1. *Using a suitable normalization, there is a unique path of homeomorphisms ψ_t, $0 \leq t < \infty$, with $\psi_t \circ g = f_t \circ \psi_{t+1}$ for rational maps f_t, so $[\psi_{t+1}] = \sigma_g([\psi_t])$. It projects to a continuous path $\pi([\psi_t])$ in moduli space. Note that $\sigma_g^n([\psi_0]) = [\psi_n]$ for $n \in \mathbb{N}$.*
2. *Suppose that $d = 2$, or more generally, that g is bicritical. Normalize the marked points $x_i(t) \in \pi([\psi_t])$ such that 0 and ∞ are critical and 1 is postcritical or marked in addition. Then the path $x_i(t)$ in moduli space is computed for $1 \leq t < \infty$ by pulling back the initial segment continuously.*

Probably the statement remains true when g is not bicritical, but the pullback is less explicit, and I am not sure if it is unique in general. Note that $[\psi_1] = \sigma_g([\psi_0])$ and an initial path ψ_t is projected to moduli space. If this condition is neglected by choosing an arbitrary path from $\pi([\psi_0])$ to $\pi([\psi_1])$, the pullback may correspond not to g but to some twisted version of it. Conditions for convergence of $\sigma_g^n([\psi_0])$ are discussed in Section 2.2; in the case of a non-(2, 2, 2, 2) orbifold, convergence in Teichmüller space is equivalent to convergence in moduli space, and in both spaces, convergence of the sequence implies convergence of the path as $t \to \infty$. The situation is more involved for an orbifold of type (2, 2, 2, 2). The implementation in terms of a piecewise linear path is discussed in [?, ?].

Proof: 1. σ_g and π are continuous. Marked points never meet under iterated pullback, so ψ_{t+1} is always defined uniquely up to Möbius conjugation.

2. In this normalization, we have $f_t(z) = m_t(z^d)$, and the Möbius transformation m_t is determined uniquely from the images of 0, 1, ∞ at time t. The path is pulled back uniquely by $f_t^{-1}(z) = \sqrt[d]{m_t^{-1}(z)}$, since any coordinate is either constant 0 or ∞, or the argument of the radical is never passing through 0 or ∞. ■

Example 2.2 (Misiurewicz polynomial mates Basilica)
*The mating of the Misiurewicz polynomial $P(z) = z^2 + \mathrm{i}$ and the Basilica polynomial $Q(z) = z^2 - 1$ is illustrated in Figure **??**. Consider the Thurston Algorithm for the formal mating g with a path according to Initialization **??** and the radius $R_t = \exp(2^{1-t})$. Rescaled to $f_t(\infty) = 1$, the initialization for $0 \leq t \leq 1$ reads*

$$x_1(t) = -\mathrm{i}/R_t^2 \quad x_2(t) = \frac{(1-\mathrm{i})/R_t^2}{1 + (1-t)\mathrm{e}^{-4}} \quad x_3(t) = \frac{\mathrm{i}/R_t^2}{1 + (1-t)2\mathrm{i}\mathrm{e}^{-4}} \,. \tag{1}$$

*Note that the normalization $x_3(t) = -x_1(t)$ is satisfied for $t \geq 1$ only. For $t \geq 0$ we have the following pullback relation, and the formula for $x_2(t+1)$ simplifies to (**??**) when $t \geq 1$:*

$$x_1(t+1) = \pm\sqrt{\frac{x_1(t) - x_2(t)}{1 - x_2(t)}} \quad x_2(t+1) = \pm\sqrt{\frac{x_1(t) - x_3(t)}{1 - x_3(t)}} \quad x_3(t+1) = -x_1(t+1) \,, \tag{2}$$

where the sign is chosen by continuity. According to Theorem **??**, *the rational maps f_t converge to the rescaled geometric mating $f(z) = (z^2+2)/(z^2-1)$, so $x_1(t) \to -2$, $x_2(t) \to 2$, and $x_3(t) \to 2$. Since two postcritical points are identified, the iteration diverges in moduli space and in Teichmüller space.*

An alternative interpretation of the path reads as follows: by a standard technique from algebraic topology, the universal cover of moduli space is constructed as the space of homotopy classes of paths with a fixed starting point. So that space is isomorphic to Teichmüller space. In this sense, the pullback of the path is a direct implementation of σ_g , and information on the dynamics of σ_g is available from homotopy classes of paths. See Section 3.3 in [**?**] for an application.

Sarah Koch [**?**] gives criteria on g for the existence of a moduli space map from $\pi(\sigma_g([\psi]))$ to $\pi([\psi])$, which is a critically finite map in the same dimension as the moduli space. See also Section 3.2 in [**?**]. Then the path may be chosen within the Julia set of the moduli space map, which is easily visualized when it is one-dimensional [**?**]. This happens for a NET map, which has four postcritical points and only simple critical points [**?**]. In the quadratic case of NET maps, a moduli space map exists if at least one critical point is postcritical, and not when g is a Lattès map of type (2, 2, 2, 2).

Example 2.3 (Obstructed self-mating)
For the self-mating of the Basilica polynomial $P(z) = Q(z) = z^2 - 1$, consider the radius $R_t = \exp(2^{1-t})$ again, and Initialization **??** *reads $x_1(t) = -1/R_t$ for $0 \le t \le 1$. The normalization is symmetric under inversion, and the pullback relation $x_1(t+1) = -\sqrt{-x_1(t)}$ has an explicit solution in this case, which is given by $x_1(t) = -1/R_t$ for $0 \le t < \infty$. So $x_1(t) \to -1$ as $t \to \infty$, and the rational maps $f_t(z) = (z^2 + x_1(t))/(1 + x_1(t)z^2)$ degenerate to a constant map. Note that there is a moduli space map $x_1(t) = -\Big(x_1(t+1)\Big)^2$, and for a different initialization, the path would be contained in the unit circle.*

3 Captures and encaptures

Captures and encaptures are ways to construct a Thurston map by shifting a critical value to a preperiodic point; we shall see that encaptures are related to matings with preperiodic polynomials in fact.
Add remarks on implementation and convergence.

These constructions rely on the concept of shifting or pushing a point from a to b along an arc C. This means that a homeomorphism φ is chosen, which is the identity outside off a tubular neighborhood of C, and such that $\varphi(a) = b$. So an unspecified point close to a is mapped to a and b is mapped to an arbitrary point nearby.

Proposition 3.1 (and definition)
Suppose P is a postcritically finite quadratic polynomial and $z_1 \in \mathcal{K}_p$ is preperiodic and not postcritical. Let the new postcritical set be $P_g = P_P \cup \{P^n(z_1)\,|\,n \ge 0\}$. Consider an arc C from ∞ to z_1 not meeting another point in P_g and choose a homeomorphism φ shifting ∞ to z_1 along C, which is the identity outside off a sufficiently small neighborhood of C. Then:

• *$g = \varphi \circ P$ is well-defined as a quadratic Thurston map with postcritical set P_g. It is a* ***capture*** *if z_1 is eventually attracting and an* ***encapture*** *in the repelling case.*

• *The combinatorial equivalence class of g depends only on P and on the homotopy class of the arc C.*

Proof: By construction, g is a postcritically finite branched cover, when the neighborhood of C does not include any postcritical point except z_1. Note that the preimages of z_1 under P are mapped to some arbitrary point by g, so if z_1 was periodic or postcritical, g would not be well-defined. Finally, if we have two different homeomorphisms φ and φ' along the same curve or along two homotopic curves, then $g' = (\varphi' \circ \varphi^{-1}) \circ g$ and the homeomorphism $\varphi' \circ \varphi^{-1}$ is isotopic to the identity, since the appended path $C' \cdot C^{-1}$ is contractible relative to $P_g \setminus \{z_1\}$. ■

Consider the following applications and possible generalizations:

- If a capture $g = \varphi \circ P$ is combinatorially equivalent to a rational map f, this gives a hyperbolic map of capture type. Let us say that f is a **Wittner capture**, if the capture path C is homotopic to a rational external ray followed by an internal ray of P; this construction is due to Ben Wittner [?] and Mary Rees [?]. Note that Rees denotes only Wittner captures as captures, while general captures are called maps of type III. Maps of this type are never matings, but they may have a representation as an anti-mating [?].

- Encaptures along external rays are related to matings in the following Section 4.

- Encaptures apply not only to polynomials P, but to rational maps in general as long as the other critical orbits are finite. This construction provides a finite **regluing** followed by a possible combinatorial equivalence. In a more general situation, a countable regluing is followed by a semi-conjugation [?, ?].

- **Recapture** means that the finite critical value $P(0)$ is shifted to a preimage of 0, resulting in a Thurston map equivalent to a hyperbolic polynomial. Relations to internal addresses and to Dehn twisted maps are discussed in [?].

Initialization 3.2 (Captures and encaptures)
Consider a capture or encapture $g = \varphi \circ P$ according to Proposition 3.1. Then the Thurston Algorithm is implemented by pulling back a path in moduli space, which is initialized as follows: normalize P such that the critical points are 0, ∞ and another point in $P_g \setminus \{z_1\}$ is 1. For $0 \le t \le 1$, $x_1(t)$ moves from ∞ to z_1 along C, while all of the other marked points stay fixed.

Under a non-conjugate-limbs condition, Wittner captures are unobstructed [?] and encaptures along external rays have only obstructions satisfying the assumptions of Theorem **??**; see below. So the sequence of rational maps converges to a rational map f, unless the orbifold of f is of type (2, 2, 2, 2): then the sequence does not converge in general, but it might converge for a special choice of C.

Proof: Note that when the preperiod of z_1 is one, the corresponding periodic point satisfies $\psi_t(-z_1) = -x_1(t)$ only for $t \ge 1$. Since $\varphi^{-1} \circ g = P \circ \mathrm{Id}$ and P is holomorphic, we have $[\mathrm{Id}] = \sigma_g([\varphi^{-1}])$ and we may initialize the Thurston Algorithm with a path ψ_t from $\psi_0 = \varphi^{-1}$ to $\psi_1 = \mathrm{Id}$. Now $\varphi^{\pm 1}$ is the identity outside off a small neighborhood of C, so ψ_t can be chosen such that it moves $x_1(t) = \psi_t(z_1)$ from $\varphi^{-1}(z_1) = \infty$ to z_1 along C, and leaves the other marked points untouched. By Proposition 2.1 the projection from $\mathcal{T}$ to $\mathcal{M}$ defines a suitable initialization to compute the Thurston pullback $\pi(\sigma_g^n)$ from an explicit pullback in moduli space. ∎

4 Encaptures and matings

The representation of matings by encaptures along external rays is motivated by remarks in [?, ?]. In the former paper, the boundary of a capture component in V_n is described by matings, which are related to the postcritically finite map of capture type by regluing. This means that the critical value is shifted from ∞ along an external ray followed by an internal ray, and then moved back along an internal ray. So can the mating be constructed by shifting the critical value directly from ∞ to $z_1 = \gamma_p(\theta)$ along the external ray $\mathcal{R}_p(\theta)$? This is true in general when z_1 is preperiodic, not only when it is on the boundary of a hyperbolic component, but we shall not discuss postcritically infinite maps here.

Theorem 4.1 (Matings as encaptures)
Suppose P is postcritically finite and θ is preperiodic, such that $q = \gamma_M(-\theta)$ is not in the conjugate limb and $z_1 = \gamma_p(\theta) \in \partial\mathcal{K}_p$ is not postcritical. Then the encapture $g_\theta = \varphi_\theta \circ P$ along $\mathcal{R}_p(\theta)$ is combinatorially equivalent or essentially equivalent to the geometric mating f defined by $P \coprod Q$.

So if $P \coprod Q$ is not of type (2, 2, 2, 2), any implementation of the Thurston pullback for g_θ gives a converging sequence of rational maps; e.g., Initialization 3.2 applies. The normalization $\beta_p = 1$ ensures $f(1) = 1$. Note that the encapture does not work if both P and Q are hyperbolic; then there is an alternative construction with two paths [?]. When only one of the two polynomials is hyperbolic, then either $P \coprod Q$ or $Q \coprod P$ is an encapture. And when both are critically preperiodic, then both $P \coprod Q$ and $Q \coprod P$ are encaptures, unless a critical point is iterated to the other critical point: then ∞ shall be iterated to 0. — By choosing encaptures along homotopic external rays, examples of shared matings are obtained in [?].

Recall the notation g and $\widetilde{g}$ for the formal mating and the essential mating; we shall see below that there is an essential encapture $\widetilde{g}_\theta$ as well. Before showing $\widetilde{g}_\theta \sim \widetilde{g}$ let us consider a few examples, to see how identifications happen and why they may happen in different ways for g and g_θ :

- When $g = 9/56 \sqcup 1/4$, so $\theta = 3/4$, there are no postcritical identifications: $\widetilde{g} = g$ and $\widetilde{g}_\theta = g_\theta$. The encapture can be constructed from the formal mating by shifting all postcritical points in $\varphi_\infty(\mathcal{K}_q)$ to $\varphi_0(\mathcal{K}_p)$ along external rays, so g_θ and g are combinatorially equivalent.

- In reverse order we have $\widetilde{g} = g = 1/4 \sqcup 9/56$ again, but $\widetilde{g}_\theta \neq g_\theta$ for $\theta = 47/56$ and $p = \gamma_M(1/4)$. Now $g_\theta(\infty)$ has preperiod and period three, but $\widetilde{g}_\theta(\infty)$ has period one. The shift φ_θ creates a subset of the lamination with angle θ in the exterior of $\mathcal{K}_p$, so there is a triangle connecting 3/7, 5/7, 6/7 with a homotopic preimage under g_θ ; pinching the surrounding Lévy-cycle gives $\widetilde{g}_\theta$.

- The converse happens for $g = 1/4 \sqcup 3/14$, so $p = \gamma_M(1/4)$ and $\theta = 11/14$. Now both $q = \gamma_M(3/14)$ and $g(\infty)$ have preperiod one and period three, while $\widetilde{g} \neq g$ has period one. But this identification is immediate in the encapture $g_\theta = \widetilde{g}_\theta$, since $z_1 = -\alpha_p$.

- Both phenomena happen at the same time for $g = 3/14 \sqcup 3/14$, so $\theta = 11/14$. In g_θ the 3-cycle of P is collapsed by a triangle in the exterior, while the 3-cycle of Q is identified with α_p immediately. We have $\widetilde{g}_\theta \neq g_\theta \not\sim g \neq \widetilde{g}$.

For longer ray connections, there may be a similar splitting of branch points and similar immediate identifications, but otherwise the encapture can be understood in terms of the same ray-equivalence classes as the formal mating:

Proof of Theorem 4.1: Denote by X the union of all postcritical ray-equivalence classes of the formal mating $g = P \sqcup Q$. Define another Thurston map g^θ by shifting the critical value $\varphi_\infty(q)$ to $\varphi_0(z_1)$ along $\mathcal{R}_\theta$, without modifying g on X. Consider the extended Hubbard tree $T_p \subset \mathcal{K}_p$, which consists of regular arcs connecting the postcritical points of g_θ. Then $g_\theta : T_p' \to T_p$, where $T_p' = T_p$ except for a slight detour at $P^{-1}(0)$. We may assume that $g^\theta \circ \varphi_0 = \varphi_0 \circ g_\theta$ in a neighborhood of T_p. So the two maps are combinatorially equivalent, even if we mark the critical point ∞ in addition, since all other marked points are contained in T_p and T_p is connected.

Now consider a path of Thurston maps g_t, such that postcritical points of P stay fixed in $\varphi_0(\partial\mathcal{K}_p)$ and all postcritical points of Q move from $\varphi_\infty(\partial\mathcal{K}_q)$ to $\varphi_0(\partial\mathcal{K}_p)$ along external rays of g. This deformation is a kind of two-sided pseudo-isotopy from g to g^θ, since marked points may collapse in different ways on both ends, while each component of X is invariant under each g_t. By collapsing all components of X to points and modification at preimages, equivalent quotient maps are obtained for all g_t, in particular for g and g^θ, where postcritical points have been identified already in different ways. So we know that $\widetilde{g}^\theta = \widetilde{g} \sim f$ and we may consider $\widetilde{g}^\theta$ as an essential map in the sense of Definition **??**, with Γ consisting of loops around those trees in X, which contain at least two postcritical points of g^θ. So g^θ is essentially equivalent to f, combinatorially equivalent if $\Gamma = \emptyset$, and the same applies to the original encapture g_θ. ■

5 Matings on the boundary of capture components

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6 Visualization of captures and encaptures

To illustrate the process of slow capture or encapture, we may also define a sequence or path of images $\psi_t(\mathcal{K}_p)$ of the filled Julia set, which is constant $\mathcal{K}_p$ for $0 \le t \le 1$. It will show more and more identifications happening by a piecewise pseudo-isotopy. See also the videos on www.mndynamics.com . A similar initialization is used for Dehn twisted maps; see [?] and the Examples 3.1 and 3.7 in [?].