Quadratic polynomials, captures, and matings

Wolf Jung
Gesamtschule Brand, 52078 Aachen, Germany.
E-mail: jung@mndynamics.com

Abstract

There is an alternative construction of mating, when at least one polynomial is preperiodic: shift the infinite critical value of the other polynomial to a preperiodic point.

1 Introduction

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2 Background

2.1 Polynomials and rational maps

2.2 The Thurston Theorem

2.3 A path in moduli space

The pullback of homeomorphisms ψ_n was easy to define, but it is not computed easily: repeated pullbacks would be defined piecewise, and solving the Beltrami equation numerically would be impractical as well. The isotopy classes in Teichmüller space are meant to represent only combinatorial information anyway: we are interested in the pullback of marked points $x_i(n) \in \pi(\sigma_g^n([\psi_0]))$ and maps f_n , and the combinatorial description is needed to make a finite choice between different possible preimages. This characterization of the topology has been implemented in terms of spiders [?, ?], medusas [?], and triangulations [?]. These contain the necessary information from Teichmüller space without using actual homeomorphisms ψ_n .

Following Bartholdi–Nekrashevych [?] and Buff–Chéritat [?], the following alternative method shall be discussed. It means that Teichmüller space is used explicitly only to check a suitable initialization of a path in moduli space. Afterward the path is pulled back simply by choosing preimages from continuity. The application to matings is discussed in Sections 7 and 8. The spider algorithm is implemented with a path in [?] and further applications to quadratic polynomials are given; twisted polynomials and Lattès maps are discussed in [?] as well.

Proposition 2.1 (Path in moduli space)
Suppose g is a Thurston map of degree $d \ge 2$, and there is a continuous path of homeomorphisms $\psi_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $0 \le t \le 1$, with $\psi_0 \circ g = f_0 \circ \psi_1$ for a rational map f_0. So $[\psi_1] = \sigma_g([\psi_0])$.
1. *Using a suitable normalization, there is a unique path of homeomorphisms ψ_t, $0 \le t < \infty$, with $\psi_t \circ g = f_t \circ \psi_{t+1}$ for rational maps f_t, so $[\psi_{t+1}] = \sigma_g([\psi_t])$. It projects to a continuous path $\pi([\psi_t])$ in moduli space. Note that $\sigma_g^n([\psi_0]) = [\psi_n]$ for $n \in \mathbb{N}$.*
2. *Suppose that $d = 2$, or more generally, that g is bicritical. Normalize the marked points $x_i(t) \in \pi([\psi_t])$ such that 0 and ∞ are critical and 1 is postcritical or marked in addition. Then the path $x_i(t)$ in moduli space is computed for $1 \le t < \infty$ by pulling back the initial segment continuously.*

Probably the statement remains true when g is not bicritical, but the pullback is less explicit, and I am not sure if it is unique in general. Note that $[\psi_1] = \sigma_g([\psi_0])$ and an initial path ψ_t is projected to moduli space. If this condition is neglected by choosing an arbitrary path from $\pi([\psi_0])$ to $\pi([\psi_1])$, the pullback may correspond not to g but to some twisted version of it. Conditions for convergence of $\sigma_g^n([\psi_0])$ are discussed in Section 2.2; in the case of a non-(2, 2, 2, 2) orbifold, convergence in Teichmüller space is equivalent to convergence in moduli space, and in both spaces, convergence of the sequence implies convergence of the path as $t \to \infty$. The situation is more involved for an orbifold of type (2, 2, 2, 2). The implementation in terms of a piecewise linear path is discussed in [**?**, **?**].

Proof: 1. σ_g and π are continuous. Marked points never meet under iterated pullback, so ψ_{t+1} is always defined uniquely up to Möbius conjugation.

2. In this normalization, we have $f_t(z) = m_t(z^d)$, and the Möbius transformation m_t is determined uniquely from the images of 0, 1, ∞ at time t. The path is pulled back uniquely by $f_t^{-1}(z) = \sqrt[d]{m_t^{-1}(z)}$, since any coordinate is either constant 0 or ∞, or the argument of the radical is never passing through 0 or ∞. ∎

Example 2.2 (Misiurewicz polynomial mates Basilica)
*The mating of the Misiurewicz polynomial $P(z) = z^2 + \mathrm{i}$ and the Basilica polynomial $Q(z) = z^2 - 1$ is illustrated in Figure **??**. Consider the Thurston Algorithm for the formal mating g with a path according to Initialization **??** and the radius $R_t = \exp(2^{1-t})$. Rescaled to $f_t(\infty) = 1$, the initialization for $0 \le t \le 1$ reads*

$$x_1(t) = -\mathrm{i}/R_t^2 \quad x_2(t) = \frac{(1-\mathrm{i})/R_t^2}{1 + (1-t)\mathrm{e}^{-4}} \quad x_3(t) = \frac{\mathrm{i}/R_t^2}{1 + (1-t)2\mathrm{i}\mathrm{e}^{-4}} \,. \tag{1}$$

*Note that the normalization $x_3(t) = -x_1(t)$ is satisfied for $t \ge 1$ only. For $t \ge 0$ we have the following pullback relation, and the formula for $x_2(t+1)$ simplifies to (**??**) when $t \ge 1$:*

$$x_1(t{+}1) = \pm\sqrt{\frac{x_1(t) - x_2(t)}{1 - x_2(t)}} \quad x_2(t{+}1) = \pm\sqrt{\frac{x_1(t) - x_3(t)}{1 - x_3(t)}} \quad x_3(t{+}1) = -x_1(t{+}1) \,, \tag{2}$$

*where the sign is chosen by continuity. According to Theorem **??**, the rational maps f_t converge to the rescaled geometric mating $f(z) = (z^2{+}2)/(z^2{-}1)$, so $x_1(t) \to -2$,*

$x_2(t) \to 2$, *and* $x_3(t) \to 2$. *Since two postcritical points are identified, the iteration diverges in moduli space and in Teichmüller space.*

An alternative interpretation of the path reads as follows: by a standard technique from algebraic topology, the universal cover of moduli space is constructed as the space of homotopy classes of paths with a fixed starting point. So that space is isomorphic to Teichmüller space. In this sense, the pullback of the path is a direct implementation of σ_g, and information on the dynamics of σ_g is available from homotopy classes of paths. See Section 3.3 in [**?**] for an application.

Sarah Koch [**?**] gives criteria on g for the existence of a moduli space map from $\pi(\sigma_g([\psi]))$ to $\pi([\psi])$, which is a critically finite map in the same dimension as the moduli space. See also Section 3.2 in [**?**]. Then the path may be chosen within the Julia set of the moduli space map, which is easily visualized when it is one-dimensional [**?**]. This happens for a NET map, which has four postcritical points and only simple critical points [**?**]. In the quadratic case of NET maps, a moduli space map exists if at least one critical point is postcritical, and not when g is a Lattès map of type (2, 2, 2, 2).

Example 2.3 (Obstructed self-mating)
For the self-mating of the Basilica polynomial $P(z) = Q(z) = z^2 - 1$, *consider the radius* $R_t = \exp(2^{1-t})$ *again, and Initialization* **??** *reads* $x_1(t) = -1/R_t$ *for* $0 \le t \le 1$. *The normalization is symmetric under inversion, and the pullback relation* $x_1(t+1) = -\sqrt{-x_1(t)}$ *has an explicit solution in this case, which is given by* $x_1(t) = -1/R_t$ *for* $0 \le t < \infty$. *So* $x_1(t) \to -1$ *as* $t \to \infty$, *and the rational maps* $f_t(z) = (z^2 + x_1(t))/(1 + x_1(t)z^2)$ *degenerate to a constant map. Note that there is a moduli space map* $x_1(t) = -\Big(x_1(t+1)\Big)^2$, *and for a different initialization, the path would be contained in the unit circle.*

3 The Thurston Algorithm for quadratic polynomials

normalization, 0 marked, asymptotics, Thurston matrix and renormalization, Hubbard tree, twisted, recapture and internal addresses

4 The spider algorithm

*In [**?**], the Thurston Algorithm with a path in moduli space is implemented for quadratic polynomials, including the spider algorithm, twisted polynomials, precapture and recapture, and slow tuning. This section sketches the discussion of the spider algorithm, because it is another application of the convergence Theorem* **??**; *in fact it was the original motivation for this research.*

For an angle $\theta \in \mathbb{Q} \setminus \mathbb{Z}$, we want to determine the associated postcritically finite parameter c of a quadratic polynomial $f_c(z) = z^2 + c$. From θ a Thurston map g_θ is constructed, and the Thurston Algorithm shall give f_c. Denote the iterates of $\theta = \theta_1$ by $\theta_i = 2^{i-1}\theta$, $i \ge 1$, and the preperiod and period of θ is k and p. Consider the map $g_\theta = \varphi_\theta \circ F$ with $F(z) = z^2$. Here the homeomorphism φ_θ is the identity in most of $\widehat{\mathbb{C}}$; it shifts the straight ray with angle θ_1 radially out by 1, and if $k = 0$, it shifts the ray with angle θ_p in by 1. So $\varphi_\theta(0) = \mathrm{e}^{\mathrm{i}2\pi\theta_1}$, and in the periodic case $\varphi_\theta(e^{\mathrm{i}2\pi\theta_p}) = 0$. The straight spider is invariant under g_θ .

To apply the Thurston Algorithm, we need to pull back marked points $x_i(n)$ with quadratic polynomials. The choice of branch for the square roots is determined by the pullback of an isotopy class of homeomorphisms. The basic idea of the **spider algorithm** is: Teichmüller space is represented by spiders, homotopy classes of graphs with legs from ∞ to the marked points, which are pulled back with the polynomials. According to [**?**, **?**, **?**] we may consider these cases:

Case 1: The angle θ is periodic and c is the center associated to the root $\gamma_M(\theta)$. Then g_θ is combinatorially equivalent to f_c and unobstructed. Under the equivalence, the spider legs are homotopic to external rays extended by internal rays, which will have common points in the satellite case.

Case 2: The angle θ is preperiodic and the Misiurewicz point $c = \gamma_M(\theta)$ is an endpoint or of primitive type. Again, g_θ is unobstructed and equivalent to f_c . The spider legs correspond to external rays at the postcritical orbit.

Case 3: The Misiurewicz point $c = \gamma_M(\theta)$ is of satellite type; the angle θ_{k+1} has period $p = rq$ and the landing point has period $q < p$. Now g_θ has a Lévy cycle with q curves, each containing r marked points. By identifying these points manually, or by extending the spider legs accordingly, a modified Thurston map $\widetilde{g}_\theta$ is defined; it is unobstructed and combinatorially equivalent to f_c .

See [**?**] for a convergence proof in the periodic case, which replaces Teichmüller space with a more explicit spider space. The essential spider map $\widetilde{g}_\theta$ is constructed in [**?**], and the relation between obstructions, kneading sequences, and the satellite case is obtained in [**?**]. Note that the description above assumes landing properties of parameter rays according to [**?**, **?**], and the spider algorithm is just a method to compute parameters numerically. Alternatively, one may discuss the spider map g_θ directly and conclude the existence of quadratic polynomials with specific combinatorics. There are several variants of implementing the spider algorithm:

- In a pullback step, each leg and endpoint has two preimages under the quadratic polynomial, or the preimage is the critical point with two legs. To choose unique preimages, either employ the cyclic order of rays at ∞, which is related to intersection numbers, or consider the angles of the legs at ∞.

- Either normalize the position of two finite marked points, or assume that all polynomials are of the form $z^2 + c_n$. This increases the dimension of Teichmüller space by one and gives an additional eigenvalue $\lambda = 1/2$.

- Each leg is encoded as a sequence of points, such that the curve is homotopic to a polygonal curve with respect to the marked points. Since the preimages of straight lines are hyperbolas in general, this means that each hyperbola segment is replaced with a line segment again; we must check that it is homotopic. When this condition is violated in the current step for one or more segments, we may either refine the discretization there (and prune somewhere else), or restart with an overall finer discretization.

In the satellite Misiurewicz case 3, Hubbard–Schleicher [?] observed that colliding marked points converge to postcritical points of f_c and the polynomials converge to f_c. To understand this process in general, Selinger [?, ?] considered the extension of the Thurston pullback to augmented Teichmüller space and the dynamics on the canonical stratum. This phenomenon motivated the research for the convergence Theorem **??** as well. Intuitively, the points must collide because the unique obstruction is pinched, and since they stay close together while moving, the pullback of g_θ shall be similar to the pullback defined by $\widetilde{g}_\theta$ or f_c. But this description involves interchanging limits, so it is not obvious that the marked points get close to the expected limit and do not jump away.

Theorem 4.1 (following Hubbard–Schleicher and Selinger)
For the pullback defined by the unmodified spider map g_θ, the polynomials converge to f_c and the marked points converge to postcritical points, with suitable collisions in the satellite Misiurewicz case 3*.*

Proof: According to the references given above, either g_θ or $\widetilde{g}_\theta$ is unobstructed and equivalent to f_c In case 3, the Thurston pullback for g_θ diverges due to the Lévy cycle. The essential map $\widetilde{g}_\theta$ is equivalent to f_c and the other component maps are homeomorphisms. So Theorem **??** applies and gives convergence immediately.

Recall the following steps of its proof. In the context of Proposition **??** the current situation was called scenario 2: the pullback in configuration space extends to a neighborhood of the prospective limit. The eigenvalues either come from the modified Thurston pullback, or they are of the form $\lambda^{rq} = \rho^{-r}$, $\lambda^q \neq \rho^{-1}$, where ρ is the repelling multiplier of the q-cycle of f_c. The techniques of Selinger show that the points in configuration space get arbitrarily close to the prospective limit, such that a segment of an invariant path in Teichmüller space projects into an attracting neighborhood of that configuration. Then it cannot happen that at some step another branch of the pullback relation becomes active, so the points do not jump away. ∎

In contrast to the situation of formal matings, this generalized convergence property is not crucial from a numerical perspective, since the modification from g_θ to $\widetilde{g}_\theta$ is simple and explicit. As a completely different approach, the parameter c may be obtained by drawing the parameter ray $\mathcal{R}_M(\theta)$ and starting a Newton iteration from the approximate endpoint. Now, let us consider an alternative implementation of the spider algorithm, which pulls back a path in moduli space instead of spiders in Teichmüller space. So the legs are invisible, but the movement of the feet is recorded:

Initialization 4.2 (Spider algorithm with a path)
Suppose $\theta = \theta_1 \in \mathbb{Q}/\mathbb{Z}$ has preperiod k and period p. Define $(x_1(t), \dots, x_{k+p}(t))$ for $0 \le t \le 1$ as

$$\begin{aligned} x_1(t) &= t \cdot \mathrm{e}^{\mathrm{i}2\pi\theta_1} & & \\ x_p(t) &= (1-t) \cdot \mathrm{e}^{\mathrm{i}2\pi\theta_p}\,, & & \textit{if} \quad k = 0 \\ x_j(t) &= \mathrm{e}^{\mathrm{i}2\pi\theta_j} & &, \quad \textit{otherwise.} \end{aligned} \tag{3}$$

Pull this path back continuously with $x_i(t+1) = \pm\sqrt{x_{i+1}(t) - x_1(t)}$. Then it converges to the marked points of f_c with appropriate collisions.

Proof: The argument is similar to that given for captures and precaptures according to Initialization 5.2, and for twisted maps according to Examples 3.1 and 3.7 in [**?**]. We may initialize the Thurston pullback for $g_\theta = \varphi_\theta \circ F$ by $\psi_0 = \varphi_\theta^{-1}$ and $\psi_1 = \mathrm{Id}$. There is an obvious deformation ψ_t along one or two rays, which projects to the path defined in moduli space. By Proposition 2.1, this shows that the pullback of the path agrees with the projection of the pullback in Teichmüller space. Note that for $k = 0$, we have $x_p(t) = 0$ only for $t \ge 1$. Likewise, for $k = 1$ the relation $x_{k+p}(t) = -x_k(t)$ is satisfied for $t \ge 1$ only. ∎

This algorithm gives the same marked points as the spider algorithm with legs, and it converges unless there are floating-point cancellations or problems with the discretization: again, the path is represented by a polygonal curve, and there is an explicit check for homotopy violations by the simultaneous deformation of hyperbola segments to line segments; if that happens, refine or restart. Since only a path of length $|n \le t \le n+1| = 1$ needs to be stored instead of full legs, we may take a large number of line segments easily, but there is a trade-off: there will be little need for refinement, because small hyperbola segments are close to small line segments, but there is a loss of precision by subtracting numbers that are approximately equal.

For exponential functions with preperiodic singular value, spiders and modified spiders are constructed in [SZ, LSV], and convergence of unobstructed pullback maps follows from [HSS]. The alternative implementation with a path in moduli space is straightforward, but a check for homotopy violations will be harder. Examples show convergence of colliding marked points analogously to Theorem 4.1. While the local analysis at the prospective limit is the same, the extension to augmented Teichmüller space is unknown and so the global analysis is incomplete.

[HSS] J. Hubbard, D. Schleicher, M. Shishikura, Exponential Thurston maps and limits of quadratic differentials, J. AMS **22**, 77–117 (2009).

[LSV] B. Laubner, D. Schleicher, V. Vicol, A Combinatorial Classification of Postsingularly Finite Complex Exponential Maps, Discrete Cont. Dyn. Systems **22**, 2008.

[SZ] D. Schleicher, J. Zimmer, Periodic points and dynamic rays of exponential maps, Ann. Acad. Scient. Fenn. **28**, 327–354 (2003).

5 Captures and encaptures

Captures and encaptures are ways to construct a Thurston map by shifting a critical value to a preperiodic point; we shall see that encaptures are related to matings with preperiodic polynomials in fact.
Add remarks on implementation and convergence.

These constructions rely on the concept of shifting or pushing a point from a to b along an arc C. This means that a homeomorphism φ is chosen, which is the identity outside off a tubular neighborhood of C, and such that $\varphi(a) = b$. So an unspecified point close to a is mapped to a and b is mapped to an arbitrary point nearby.

Proposition 5.1 (and definition)
Suppose P is a postcritically finite quadratic polynomial and $z_1 \in \mathcal{K}_p$ is preperiodic and not postcritical. Let the new postcritical set be $P_g = P_P \cup \{P^n(z_1) \,|\, n \ge 0\}$. Consider an arc C from ∞ to z_1 not meeting another point in P_g and choose a homeomorphism φ shifting ∞ to z_1 along C, which is the identity outside off a sufficiently small neighborhood of C. Then:

• *$g = \varphi \circ P$ is well-defined as a quadratic Thurston map with postcritical set P_g. It is a* ***capture*** *if z_1 is eventually attracting and an* ***encapture*** *in the repelling case.*

• *The combinatorial equivalence class of g depends only on P and on the homotopy class of the arc C.*

Proof: By construction, g is a postcritically finite branched cover, when the neighborhood of C does not include any postcritical point except z_1. Note that the preimages of z_1 under P are mapped to some arbitrary point by g, so if z_1 was periodic or postcritical, g would not be well-defined. Finally, if we have two different homeomorphisms φ and φ' along the same curve or along two homotopic curves, then $g' = (\varphi' \circ \varphi^{-1}) \circ g$ and the homeomorphism $\varphi' \circ \varphi^{-1}$ is isotopic to the identity, since the appended path $C' \cdot C^{-1}$ is contractible relative to $P_g \setminus \{z_1\}$. ■

Consider the following applications and possible generalizations:

- If a capture $g = \varphi \circ P$ is combinatorially equivalent to a rational map f, this gives a hyperbolic map of capture type. Let us say that f is a **Wittner capture**, if the capture path C is homotopic to a rational external ray followed by an internal ray of P; this construction is due to Ben Wittner [**?**] and Mary Rees [**?**]. Note that Rees denotes only Wittner captures as captures, while general captures are called maps of type III. Maps of this type are never matings, but they may have a representation as an anti-mating [**?**].

- Encaptures along external rays are related to matings in the following Section 6.

- Encaptures apply not only to polynomials P, but to rational maps in general as long as the other critical orbits are finite. This construction provides a finite **regluing** followed by a possible combinatorial equivalence. In a more general situation, a countable regluing is followed by a semi-conjugation [**?**, **?**].

- **Recapture** means that the finite critical value $P(0)$ is shifted to a preimage of 0, resulting in a Thurston map equivalent to a hyperbolic polynomial. Relations to internal addresses and to Dehn twisted maps are discussed in [**?**].

Initialization 5.2 (Captures and encaptures)
Consider a capture or encapture $g = \varphi \circ P$ according to Proposition 5.1. Then the Thurston Algorithm is implemented by pulling back a path in moduli space, which is initialized as follows: normalize P such that the critical points are 0, ∞ and another point in $P_g \setminus \{z_1\}$ is 1. For $0 \le t \le 1$, $x_1(t)$ moves from ∞ to z_1 along C, while all of the other marked points stay fixed.

Under a non-conjugate-limbs condition, Wittner captures are unobstructed [**?**] and encaptures along external rays have only obstructions satisfying the assumptions of Theorem **??**; see below. So the sequence of rational maps converges to a rational map f, unless the orbifold of f is of type $(2, 2, 2, 2)$: then the sequence does not converge in general, but it might converge for a special choice of C.

Proof: Note that when the preperiod of z_1 is one, the corresponding periodic point satisfies $\psi_t(-z_1) = -x_1(t)$ only for $t \ge 1$. Since $\varphi^{-1} \circ g = P \circ \mathrm{Id}$ and P is holomorphic, we have $[\mathrm{Id}] = \sigma_g([\varphi^{-1}])$ and we may initialize the Thurston Algorithm with a path ψ_t from $\psi_0 = \varphi^{-1}$ to $\psi_1 = \mathrm{Id}$. Now $\varphi^{\pm 1}$ is the identity outside off a small neighborhood of C, so ψ_t can be chosen such that it moves $x_1(t) = \psi_t(z_1)$ from $\varphi^{-1}(z_1) = \infty$ to z_1 along C, and leaves the other marked points untouched. By Proposition 2.1 the projection from $\mathcal{T}$ to $\mathcal{M}$ defines a suitable initialization to compute the Thurston pullback $\pi(\sigma_g^n)$ from an explicit pullback in moduli space. ■

6 Encaptures and matings

The representation of matings by encaptures along external rays is motivated by remarks in [**?**, **?**]. In the former paper, the boundary of a capture component in V_n is described by matings, which are related to the postcritically finite map of capture type by regluing. This means that the critical value is shifted from ∞ along an external ray followed by an internal ray, and then moved back along an internal ray. So can the mating be constructed by shifting the critical value directly from ∞ to $z_1 = \gamma_p(\theta)$ along the external ray $\mathcal{R}_p(\theta)$? This is true in general when z_1 is preperiodic, not only when it is on the boundary of a hyperbolic component, but we shall not discuss postcritically infinite maps here.

Theorem 6.1 (Matings as encaptures)
Suppose P is postcritically finite and θ is preperiodic, such that $q = \gamma_M(-\theta)$ is not in the conjugate limb and $z_1 = \gamma_p(\theta) \in \partial\mathcal{K}_p$ is not postcritical. Then the encapture $g_\theta = \varphi_\theta \circ P$ along $\mathcal{R}_p(\theta)$ is combinatorially equivalent or essentially equivalent to the geometric mating f defined by $P \coprod Q$.

So if $P \coprod Q$ is not of type (2, 2, 2, 2), any implementation of the Thurston pullback for g_θ gives a converging sequence of rational maps; e.g., Initialization 5.2 applies. The normalization $\beta_p = 1$ ensures $f(1) = 1$. Note that the encapture does not work if both P and Q are hyperbolic; then there is an alternative construction with two paths [**?**]. When only one of the two polynomials is hyperbolic, then either $P \coprod Q$ or $Q \coprod P$ is an encapture. And when both are critically preperiodic, then both $P \coprod Q$ and $Q \coprod P$ are encaptures, unless a critical point is iterated to the other critical point: then ∞ shall be iterated to 0. — By choosing encaptures along homotopic external rays, examples of shared matings are obtained in [**?**].

Recall the notation g and $\widetilde{g}$ for the formal mating and the essential mating; we shall see below that there is an essential encapture $\widetilde{g}_\theta$ as well. Before showing $\widetilde{g}_\theta \sim \widetilde{g}$ let us consider a few examples, to see how identifications happen and why they may happen in different ways for g and g_θ :

- When $g = 9/56 \sqcup 1/4$, so $\theta = 3/4$, there are no postcritical identifications: $\widetilde{g} = g$ and $\widetilde{g}_\theta = g_\theta$. The encapture can be constructed from the formal mating by shifting all postcritical points in $\varphi_\infty(\mathcal{K}_q)$ to $\varphi_0(\mathcal{K}_p)$ along external rays, so g_θ and g are combinatorially equivalent.

- In reverse order we have $\widetilde{g} = g = 1/4 \sqcup 9/56$ again, but $\widetilde{g}_\theta \neq g_\theta$ for $\theta = 47/56$ and $p = \gamma_M(1/4)$. Now $g_\theta(\infty)$ has preperiod and period three, but $\widetilde{g}_\theta(\infty)$ has period one. The shift φ_θ creates a subset of the lamination with angle θ in the exterior of $\mathcal{K}_p$, so there is a triangle connecting 3/7, 5/7, 6/7 with a homotopic preimage under g_θ ; pinching the surrounding Lévy-cycle gives $\widetilde{g}_\theta$.

- The converse happens for $g = 1/4 \sqcup 3/14$, so $p = \gamma_M(1/4)$ and $\theta = 11/14$. Now both $q = \gamma_M(3/14)$ and $g(\infty)$ have preperiod one and period three, while $\widetilde{g} \neq g$ has period one. But this identification is immediate in the encapture $g_\theta = \widetilde{g}_\theta$, since $z_1 = -\alpha_p$.

- Both phenomena happen at the same time for $g = 3/14 \sqcup 3/14$, so $\theta = 11/14$. In g_θ the 3-cycle of P is collapsed by a triangle in the exterior, while the 3-cycle of Q is identified with α_p immediately. We have $\widetilde{g}_\theta \neq g_\theta \not\sim g \neq \widetilde{g}$.

For longer ray connections, there may be a similar splitting of branch points and similar immediate identifications, but otherwise the encapture can be understood in terms of the same ray-equivalence classes as the formal mating:

Proof of Theorem 6.1: Denote by X the union of all postcritical ray-equivalence classes of the formal mating $g = P \sqcup Q$. Define another Thurston map g^θ by shifting the critical value $\varphi_\infty(q)$ to $\varphi_0(z_1)$ along $\mathcal{R}_\theta$, without modifying g on X. Consider the extended Hubbard tree $T_p \subset \mathcal{K}_p$, which consists of regular arcs connecting the postcritical points of g_θ. Then $g_\theta : T_p' \to T_p$, where $T_p' = T_p$ except for a slight detour at $P^{-1}(0)$. We may assume that $g^\theta \circ \varphi_0 = \varphi_0 \circ g_\theta$ in a neighborhood of T_p. So the two maps are combinatorially equivalent, even if we mark the critical point ∞ in addition, since all other marked points are contained in T_p and T_p is connected.

Now consider a path of Thurston maps g_t, such that postcritical points of P stay fixed in $\varphi_0(\partial\mathcal{K}_p)$ and all postcritical points of Q move from $\varphi_\infty(\partial\mathcal{K}_q)$ to $\varphi_0(\partial\mathcal{K}_p)$ along external rays of g. This deformation is a kind of two-sided pseudo-isotopy from g to g^θ, since marked points may collapse in different ways on both ends, while each component of X is invariant under each g_t. By collapsing all components of X to points and modification at preimages, equivalent quotient maps are obtained for all g_t, in particular for g and g^θ, where postcritical points have been identified already in different ways. So we know that $\widetilde{g}^\theta = \widetilde{g} \sim f$ and we may consider $\widetilde{g}^\theta$ as an essential map in the sense of Definition **??**, with Γ consisting of loops around those trees in X, which contain at least two postcritical points of g^θ. So g^θ is essentially equivalent to f, combinatorially equivalent if $\Gamma = \emptyset$, and the same applies to the original encapture g_θ. ■

7 Matings on the boundary of capture components

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8 Visualization of captures and encaptures

To illustrate the process of slow capture or encapture, we may also define a sequence or path of images $\psi_t(\mathcal{K}_p)$ of the filled Julia set, which is constant $\mathcal{K}_p$ for $0 \le t \le 1$. It will show more and more identifications happening by a piecewise pseudo-isotopy. See also the videos on www.mndynamics.com . A similar initialization is used for Dehn twisted maps; see [**?**] and the Examples 3.1 and 3.7 in [**?**].