Quadratic matings: Thurston Algorithm & combinatorics

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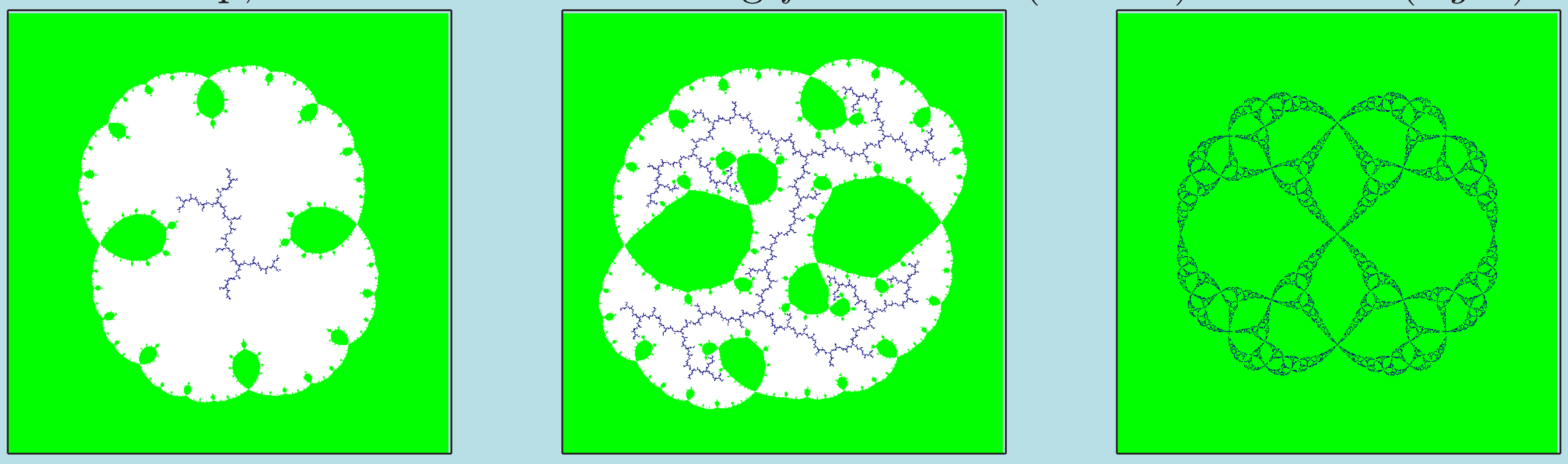
Definitions of mating

Combine two quadratic polynomials to obtain a rational map. Classical results by Douady–Hubbard and Rees–Shishikura–Tan.
Topological mating: glue filled Julia sets of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$.
Geometric mating: rational map conjugate to the topological mating.
Formal mating g: planes of polynomials are identified with half-spheres (*left*).
In the postcritically finite case, the Thurston Algorithm defines an equivalent rational map, the combinatorial mating f. Iteration (*middle*) and limit (*right*).



Actually, in the example of $p = \mathrm{i}$ and $q = -1$, the iteration diverges in Teichmüller space and in moduli space, because two postcritical points of P are identified in the limit. But the rational maps do converge.

Slow mating algorithm

The pullback of marked points in moduli space requires a choice of square-roots. It is determined by combinatorial–topological data in Teichmüller space, which have been implemented with Medusas (Hubbard et alii, Boyd–Henriksen) or triangulations (Bartholdi).
The slow mating algorithm pulls back a path in moduli space, where the choice of square-root is determined from continuity (Bartholdi–Nekrashevych, Buff–Chéritat). Teichmüller space is used only to define a suitable initialization, which is given by simple formulas involving an initial radius $R > 2$. For large R, slow mating approximates equipotential gluing, an alternative definition of mating (Milnor, Petersen–Meyer, Chéritat, Buff–Epstein–Koch).

Convergence of slow mating

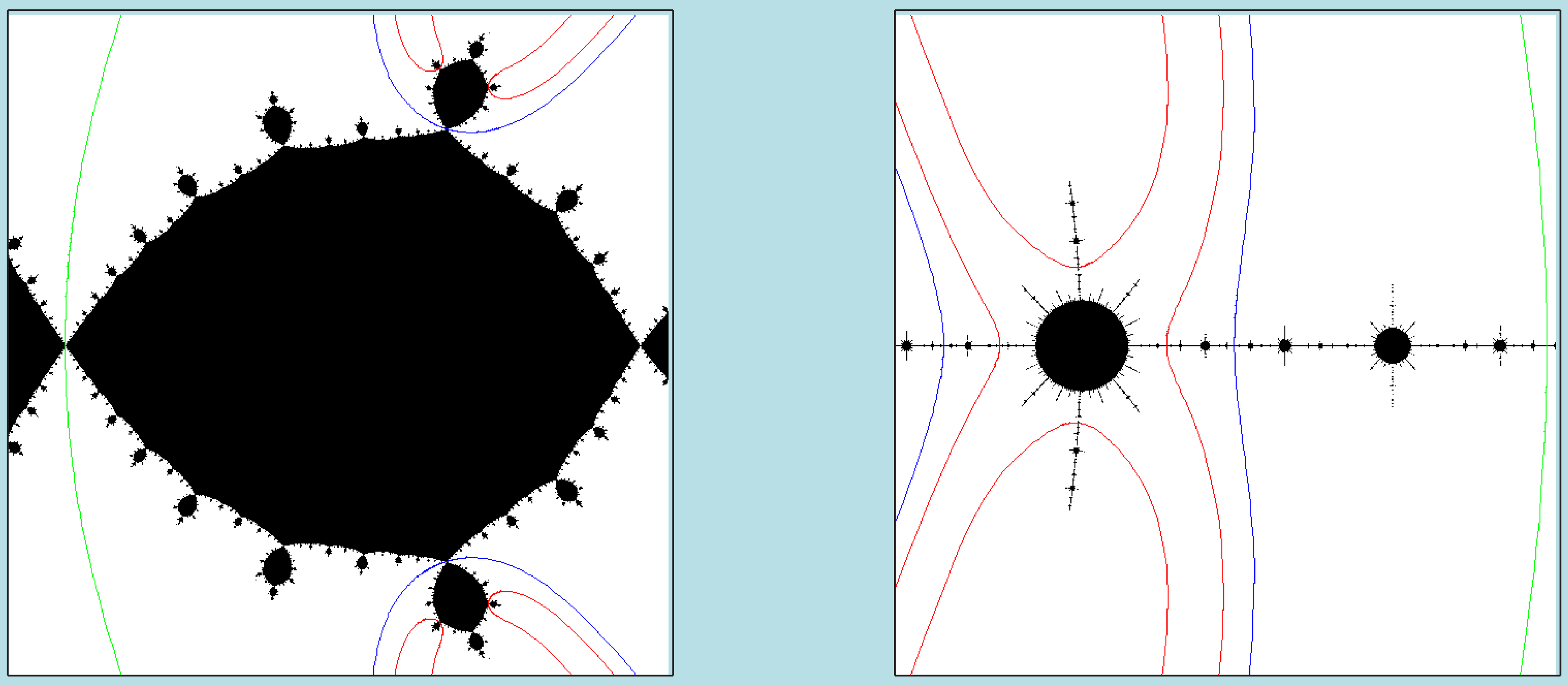
The Thurston Algorithm of g is divergent, when two postcritical points belong to the same ray-equivalence class, since they need to be identified. This can be done by modifying g to an essential mating $\widetilde{g}$ (Rees, Shishikura). Alternatively:

Theorem 1 (Convergence of maps and rational ray-equivalence classes)
When the unmodified Thurston map $g = P \sqcup Q$ has removable obstructions, the rational maps do converge to the combinatorial mating f in a suitable normalization, at least when the orbifold is hyperbolic. All rational ray-equivalence classes are collapsed, and converge to (pre-)periodic points of f.

- So we can implement the unmodified Thurston Algorithm without caring about the topology of postcritical ray-equivalence classes.
- Implications on convergence of Julia sets and holomorphic motions.
- The proof is based on Selinger's extension of the pullback to augmented Teichmüller space, as conjectured by Boyd–Henriksen.

Hausdorff obstructions

Theorem 2 (Unbounded cyclic ray connections)
Suppose p primitive renormalizable and $\mathcal{K}_p$ locally connected. There are parameters $c_* \prec c_0 \prec p$, such that for all parameters q with $\overline{q}$ on the open arc from c_* to c_0, the formal mating $g = P \sqcup Q$ has non-uniformly bounded cyclic ray connections. Moreover, these are nested such that the ray-equivalence relation is not closed. (Airplane $\sqcup$ Basilica is due independently to Bartholdi–Dudko.)



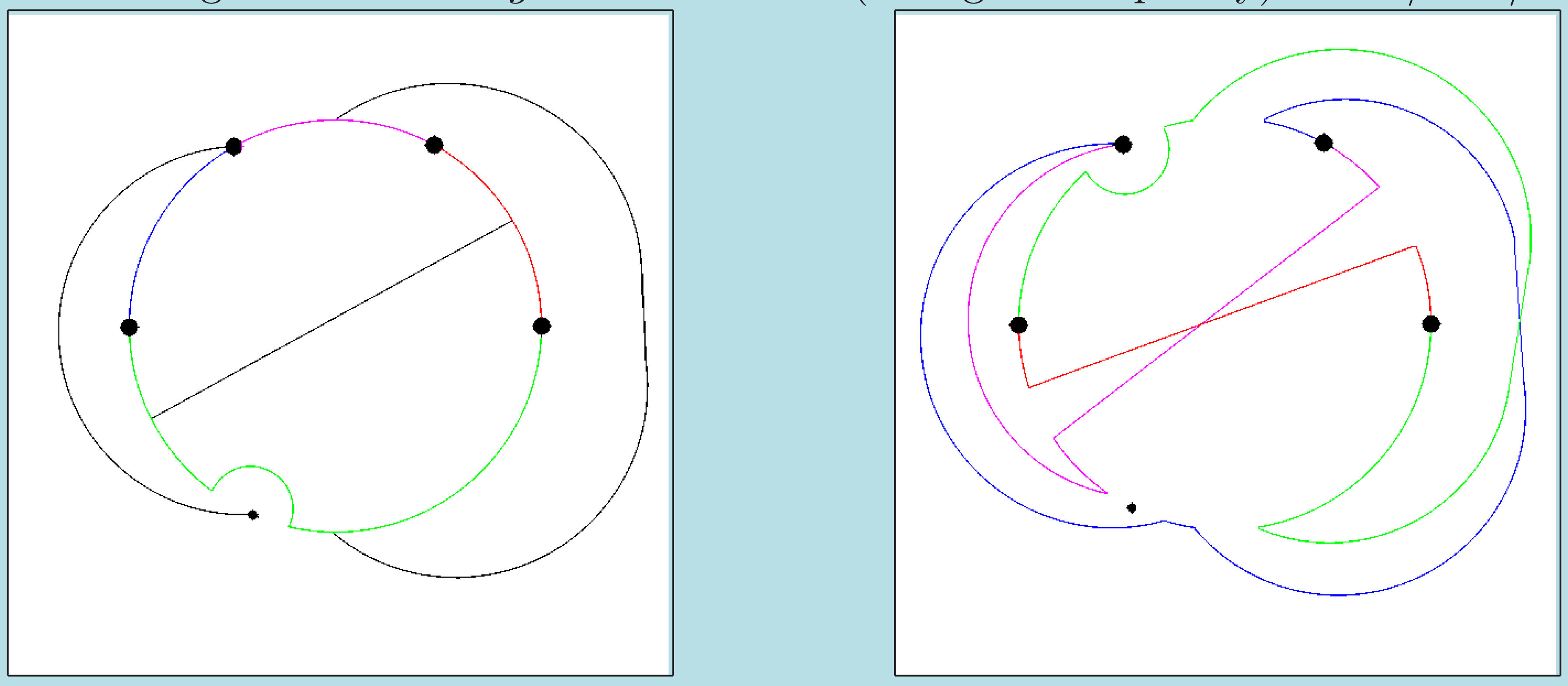
Lattès matings

Lattès maps f are double covered by an affine map $L(w) = \eta w + \kappa$ on a torus. Seven of the following matings are due to Shishikura; the two other ones of case a) answer a question of Milnor: the Peano curve γ is not unique.

	$L(w) = \eta w + \kappa$	geometric mating	anti-mating
a)	$\kappa = 0,\ \eta^2 = 2\mathrm{i}$	$f \cong 3/4 \coprod 3/4$ $f \simeq 5/28 \coprod 13/28$ $f \simeq 7/60 \coprod 29/60$	$f \cong 1/4 \coprod 1/4$
b)	$\kappa = 0,\ \eta^2 = -2$	$f \simeq 1/12 \coprod 5/12$	—
c)	$\kappa = 1/2$, $\eta^2 = \frac{-3+\mathrm{i}\sqrt{7}}{2}$	$f \cong 5/6 \coprod 5/6$	$f \cong 3/14 \coprod 3/14$
d)	$\kappa = 0$, $\eta^2 = \frac{-3+\mathrm{i}\sqrt{7}}{2}$	$f \cong 1/6 \coprod 5/14$ $f \cong 3/14 \coprod 3/14$ $f \simeq 3/14 \coprod 1/2$ $f \simeq 5/6 \coprod 1/2$	$f \cong 5/6 \coprod 5/6$

Theorem 3 (Lattès matings)
1. *There are precisely 30 formal matings $g = P \sqcup Q$ of quadratic polynomials, such that the essential mating $\widetilde{g}$ has a parabolic orbifold of type $(2, 2, 2, 2)$, and the parameters p and q are not in conjugate limbs of the Mandelbrot set. Up to complex conjugation and interchanging P and Q, these matings are represented by the nine matings in the table.*
2. *In each case, the essential mating is Thurston-equivalent to a rational map $f \simeq P \coprod Q$, which is described by η^2 in the table.*

The proof of item 1 is based on polynomial combinatorics and Sharland's observation that a fixed ray-equivalence class must contain a polynomial fixed point. Item 2 is obtained from the Shishikura Algorithm: represent the essential mating by a lamination, lift its pullback to a lattice to obtain an equivalent affine map, compute the eigenvalue η. Note that in the exceptional case of type $(2, 2, 2, 2)$, it is not enough to show that $\widetilde{g}$ is unobstructed (Selinger–Yampolsky). For $1/6 \sqcup 1/2$:



Divergence of slow mating

The Thurston pullback of a Lattès map has a neutral fixed point. For the formal matings above, the pullback of marked points has attracting multipliers from pinching removable obstructions, and an attracting center manifold. So:

Theorem 4 (Divergence of Lattès matings)
The slow mating algorithm is divergent when f is of type $(2, 2, 2, 2)$, except for $\pm 1/4 \sqcup \pm 1/4$ due to its symmetric initialization.

This is joint work with Arnaud Chéritat; please watch his movie of $1/6 \sqcup 1/6$.

Bounded ray connections

Suppose $\mathcal{K}_p$ and $\mathcal{K}_q$ are locally connected, with p in the 1/3-limb of $\mathcal{M}$, e.g., and q in the Airplane component or before it. Now there are no direct ray connections between the Hubbard tree $T_{\overline{q}} \subset \mathcal{K}_{\overline{q}}$ and one side of the arc $[\alpha_p, -\alpha_p] \subset \mathcal{K}_p$.

Theorem 5 (Examples of matings with bounded ray connections)
Then all ray-equivalence classes of the formal mating $P \sqcup Q$ are uniformly bounded trees, and the topological mating $P \coprod Q$ exists as a branched cover.

- When P and Q are geometrically finite, this provides a construction independent of the Thurston and Rees–Shishikura–Tan Theorems. A question of Epstein.
- When P or Q is not geometrically finite, probably the topological mating has not been constructed by other methods, except for $q = -1$, but the geometric mating is not constructed here either.